Example 6. Find the sum of the series

$$\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots .$$

Solution. This is a geometric series $a + ax + ax^2 + \cdots$ in which $a = \left(\frac{1}{2}\right)^3$. We simply factor a out:

$$\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^5 + \cdots + = \left(\tfrac{1}{2}\right)^3 \left[1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots\right]$$

$$= \left(\tfrac{1}{2}\right)^3 \tfrac{1}{1-1/2} = \tfrac{1}{4}. \ \ddagger$$

Rule (10) applies only to convergent series. But there is a related idea which is often useful in deciding a series diverges. It is this:

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ diverges and } c \neq 0, \text{ then } \sum_{n=1}^{\infty} c a_n \text{ diverges.} \tag{11}$$

Thus a known divergent series can't be turned into a convergent series by multiplying each term by a constant, unless that constant is zero. For each partial sum of the second series is the product of c and the corresponding partial sum of the first series; therefore if one diverges so must the other.

Example 7.

(a) $$\sum_{n=1}^{\infty} \tfrac{5}{n} \text{ diverges because } \sum_{n=1}^{\infty} \tfrac{1}{n} \text{ diverges.}$$

(b) $$\sum_{n=1}^{\infty} \tfrac{3n}{n+1} \text{ diverges because } \sum_{n=1}^{\infty} \tfrac{n}{n+1} \text{ diverges. } \ddagger$$

To this point we have used only series starting with a_0 or a_1, but it is also sometimes necessary to use series that begin with a_k:

$$\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots .$$

Indeed, any convergent series can be written as the sum of an initial block of terms plus another convergent series. Thus,

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n,$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n,$$

and, in general,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_k + \sum_{n=k+1}^{\infty} a_n \tag{12}$$

for each positive integer k. This formula, an easy consequence of comparing partial sums of the two series, makes no sense for divergent series.

Example 8.

(a) $$\sum_{n=1}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n \text{ if } |x| < 1.$$

(b) $$\sum_{n=1}^{\infty} \tfrac{n}{n(n+1)} = \tfrac{1}{2} + \tfrac{1}{6} + \sum_{n=3}^{\infty} \tfrac{1}{n(n+1)}.$$

Example 9. Find the sum of the series $\displaystyle\sum_{k=3}^{\infty} 1/2^k$.

By (12),

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2^2} + \sum_{k=3}^{\infty} \frac{1}{2^k};$$

also

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1 - 1/2} = 2.$$

Therefore,

$$\begin{aligned} \sum_{k=3}^{\infty} \tfrac{1}{2^n} &= \sum_{k=0}^{\infty} \tfrac{1}{2^k} - 1\tfrac{1}{2} - \tfrac{1}{2^2} \\ &= 2 - \tfrac{7}{4} = \tfrac{1}{4}. \end{aligned}$$

‡

Whether a series converges or diverges has nothing to do with where the indexing starts. This simple observation saves considerable fussing in practice. Specifically, if the series $\displaystyle\sum_{n=k}^{\infty} a_n$ converges for some starting index k, it will converge for any other starting index; and if $\displaystyle\sum_{n=k}^{\infty} a_n$ diverges for some starting index, it will diverge for any other starting index. To illustrate, since

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$$

is a convergent series, so is the series

$$\sum_{n=100}^{\infty} \frac{1}{2^n} = \frac{1}{2^{100}} + \frac{1}{2^{101}} + \frac{1}{2^{102}} + \cdots;$$

and since the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges, the series

$$\sum_{n=1}^{\infty} \frac{1}{n+5} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

is also divergent.

4. Series of Positive Terms

More often than not the question of convergence for an infinite series cannot be settled with the ease suggested by the preceding section, the difficulty being that it is not usually possible to find a simple formula for the nth partial sums $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ of the series $\sum a_n$. Consequently, the technique of establishing convergence by directly finding the limit of the sequence s_n must be abandoned in favor of various indirect approaches. The most basic of these methods of *testing* a series for convergence apply to series in which all the terms are positive. For brevity we shall refer to a series $\sum a_n$ in which $a_n > 0$ as a *series of positive terms*. For such a series we always have

$$s_n = s_{n-1} + a_n > s_{n-1}$$

so that the partial sums form a strictly increasing sequence. If the partial sums are also bounded, that is, if there is a constant M such that $s_n \leq M$ for all n, then the series must converge because, by Theorem 5, every increasing and bounded sequence has a limit. But if the partial sums are not bounded, then the series diverges. These remarks establish a basic principle about series of positive terms.

Theorem 10. A series $\sum a_n$ of positive terms converges if its partial sums are bounded, and diverges if they are not.

In many series the nth term a_n is the value of a simple function f at n,

$$a_n = f(n)$$

and it becomes fruitful to analyze the partial sums of

$$a_1 + a_2 + a_3 + \cdots = f(1) + f(2) + f(3) + \cdots$$

by comparing areas. The comparison is particularly simple if the associated function f is continuous, positive, and decreasing.

Theorem 11. The Integral Test

Let f be continuous, positive, and decreasing on the interval $x \geq 1$. Then the series

$$f(1) + f(2) + f(3) + \cdots \tag{1}$$

and the improper integral

$$\int_1^\infty f(x)\,dx \tag{2}$$

either both converge or both diverge.

Proof. Since f is decreasing on each of the unit intervals $n \leq x \leq n+1$, the area under the graph from n to $n+1$ is less than that of a rectangle of area $f(n)$ (Figure 14.6) and greater than that of a rectangle of area $f(n+1)$ (Figure 14.5). In particular,

$$f(2) < \int_1^2 f(x)\,dx < f(1),$$

$$f(3) < \int_2^3 f(x)\,dx < f(2),$$

$$f(4) < \int_3^4 f(x)\,dx < f(3),$$

$$f(n+1) < \int_n^{n+1} f(x)\,dx < f(n),$$

Addition yields

$$f(2) + f(3) + \cdots + f(n+1) < \int_1^{n+1} f(x)\,dx < f(1) + f(2) + \cdots + f(n).$$

In terms of the partial sums $s_n = \sum_{k=1}^{n} f(k)$, these two inequalities state that

$$(3) \qquad s_{n+1} - f(1) < \int_1^{n+1} f(x)\,dx \text{ and } \int_1^{n+1} f(x)\,dx < s_n$$

These comparisons form the basis of the integral test.

The second inequality shows that if the improper integral (2) diverges, that is, if $\int_1^{n+1} f(x)\,dx \to \infty$ as $n \to \infty$, then $s_n \to \infty$ as $n \to \infty$ and the series diverges. But if the improper integral converges, then $\int_1^{n+1} f(x)\,dx$ increases to a limit L as $n \to \infty$ and

$$s_{n+1} < f(1) + \int_1^{n+1} f(x)\,dx < f(1) + L$$

for all n. Hence $\{s_n\}$, being increasing and bounded, has a limit by Theorem 10. ‡

As a first application of the integral test, consider the harmonic series $\sum 1/n$, a known divergent series. Clearly, the test function is $f(x) = 1/x$ because $f(n) = 1/n$. And since $1/x$ is continuous, positive and decreasing on the range $x > 0$, we conclude from

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln\, x]_1^b = \lim_{b\to\infty} \ln\, b = \infty$$

that the harmonic series diverges. Moreover, the second inequality in (3) provides a rough estimate of the rate at which its partial sums grow:

$$\ln\,(n+1) = \int_1^{n+1} \frac{1}{x}\,dx < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = s_n.$$

It is interesting to compare the estimates given by this inequality with some computed values of s_n for selected n:

Estimates	Computed Values
$s_{83} > 4.4308$	$s_{83} = 5.002068...$
$s_{227} > 5.4293$	$s_{227} = 6.004366...$
$s_{616} > 6.4248$	$s_{616} = 7.001274...$
$s_{1,674} > 7.4235$	$s_{1,674} = 8.000485...$
$s_{4,550} > 8.4231$	$s_{4,550} = 9.000208...$
$s_{12,367} > 9.4228$	$s_{12,367} = 10.000043...$

The harmonic series is a special case of the more general *p-series*

(4) $$\sum_{n=1}^{\infty} \tfrac{1}{n^p} = 1 + \tfrac{1}{2^p} + \tfrac{1}{3^p} + \cdots .$$

Example 1. Show that the p-series converges if $p > 1$ and diverges if $p \leq 1$.

Solution. Here the test function is $f(x) = 1/x^p$ because $f(n) = 1/n^p$. It is continuous, positive, and since $f'(x) = -p/x^{p+1} < 0$, it is decreasing on the interval $x > 0$. If $p = 1$, the series (4) is the divergent harmonic series. If $p \neq 1$, the p-series converges or diverges with the improper integral

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1 b\, x^{-p}\, dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b = \lim_{b\to\infty} \left(\frac{b^{-p+1}}{1-p} - \frac{1}{1-p} \right).$$

Whether this limit is finite or infinite depends on the exponent $-p+1$. If $-p+1 > 0$ or $p < 1$, b^{-p+1} will increase without bound as b increases; but if $-p+1 < 0$, the exponent on b is negative and $b^{-p+1} \to 0$ as $b \to \infty$. Hence

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \frac{1}{p-1} \text{ if } p > 1 \text{ and } \int_1^{\infty} \frac{1}{x^p}\, dx \text{ diverges if } p < 1.$$

According to the integral test this means that the p-series (4) converges if $p > 1$ and diverges if $0 < p < 1$. ‡

In particular, this means that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^4}$$

are convergent series ($p = 2$ and $p = 4$) while

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

are divergent series ($p = 1/2$ and $p = 1/3$). the conclusions appear reasonable because, for instance, $1/n^2$ certainly goes to 0 much faster than $1/\sqrt{n}$. But, intuitively, no meaningful distinction is apparent between the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.001}} \text{and } \sum_{n=1}^{\infty} \frac{1}{n^{.999}}.$$