CHAPTER 1

Infinite Series

As we saw in the preceding chapter, a smooth function can be approximated by its Taylor series expansion. And our experience, based on examples, is that the accuracy of such approximations improves as the number of terms increase. The idea of approximation by finite series is a truncated version of the more general idea of representation by an infinite series. In this chapter we shall consider the elementary theory of infinite series in some detail and, in the later sections, discuss the representation of functions by special infinite series called power series.

1. Infinite Sequences

At the intuitive level a *sequence* is an ordered succession of real numbers such as

$$1, 3, 5, 7 \qquad\qquad 1, 5, 14, 30, 55$$

and

$$1, 2, 3, ..., n... \qquad \text{or} \qquad 1^2, (1/2)^2, (1/3)^2, ..., (1/n)^2, ...$$

The first two sequences stop; they are *finite sequences*. The second two never stop; they are called *infinite sequences*. Only infinite sequences are of interest to us and therefore we shall use "sequence" to mean "infinite sequence". The members of a sequence are called *terms* and they are ordered in the sense that there is a *first term*, a *second term*, a *third term*, and so forth. It is customary to denote the nth term of a sequence by a subscripted symbol such as a_n, b_n, x_n, or y_n and indicate the sequence by writing

$$a_1, a_2, a_3, ..., a_n... \, . \tag{1}$$

The sequence (1) can be defined by the compact notation $\{a_n\}$. Thus $\{n^2\}$ denotes the sequence

$$1^2, 2^2, 3^2, ..., n^2, ...,$$

and $\{\sin\ n\pi/2\}$ denotes the sequence

$$\sin\ \frac{\pi}{2},\ \sin\ \frac{2\pi}{2},\ \sin\ \frac{3\pi}{2}, ..., \frac{n\pi}{2}$$

In special situations we may find it convenient not to label the first term of a sequence with the first counting number. In such instances we adopt the following notation:

$$\{a_n\}_{n=0}^{\infty} \text{ to mean } a_0, a_1, a_2, ..., a_n, ...$$

$$\{a_n\}_{n=2}^{\infty} \text{ to mean } a_2, a_3, a_4, ..., a_n, ...$$

and

$$\{a_n\}_{n=k}^{\infty}\text{to mean } a_k, a_{k+1}, a_{k+2}, ..., a_n, ...$$

From the notation it is clear that a sequence is a special type of function in which the independent variable ranges over an infinite block of integers. This observation prompts the following definition.

Definition

A *sequence* is a function whose domain is the set of all integers greater than or equal to some fixed integer k. The value of a sequence at n is called the *nth term*.

Sequences are introduced in a variety of ways. Usually the terms are computed from a formula for a_n. The following table illustrates this situation for a number of simple examples:

nth term	first five terms	100th term
$a_n = 2n + 1$	3,5,7,9,11	201
$a_n = 1 + (-1)^n$	0,2,0,2,0	2
$a_n = \dfrac{(-1)^n n^2}{4n-1}$	$-1/3, 4/7, -9/11, 16/15, -25/19$	$10,000/399$
$a_n = n/(n+1)$	$1/2,\ 2/3,\ 3/4,\ 4/5,\ 5/6$	$100/101$

In each case the terms are obtained by substituting the values $n = 1, 2, 3, 4, 5, 100$. Of course, a_n need not be an integer or a simple fraction. And in many instances, such as

$$a_n = n\int_0^n e^{t^2}\,dt \text{ or } a_n = \sin\left(e^{\cos n}\right),$$

it may not be possible to calculate each term exactly. Occasionally a_n is not given by a single formula, as in the rule

$$a_n = \begin{cases} n^2 & \text{if } n \text{ is even} \\ \pi & \text{if } n \text{ is odd} \end{cases}$$

which defines the sequence

$$\pi, 4, \pi, 16, \pi, 36, ...$$

And there is the possibility of not being able to write any kind of formula for a_n. This is certainly the case for the sequence $\{a_n\}$ in which a_n is the nth digit in the decimal expansion of the irrational number $\sqrt{2}$. Finally, it may be that each term of a sequence is defined in terms of its predecessors as, for example,

$$a_{n+1} = a_n + \frac{1}{a_n},\ a_1 = 1.$$

The first few terms are

$$a_1 = 1,\ a_2 = 2,\ a_3 = 5/2,\ a_4 = 20/10,\ a_5 = 941/290,\ a_6 = 969,581/272,890$$

Given in this manner, a sequence is said to be defined *recursively*. Newton's formula $x_{n+1} = x_n - f(x_n)/f'(x_n),\ n = 0, 1, 2, ...,$ for finding a root of $f(x) = 0$ is a recursively defined sequence. For example, using this formula to find the root of $x^3 - 4 = 0$ leads to the recursive sequence $x_{n+1} = \frac{2}{3}[x_n + 2/x_n^2]$, $n = 0, 1, 2, ...$ whose terms, starting with, say, $x_0 = 2$, will approximate the cube root of 4.

We are interested in studying sequences $\{a_n\}$ with the property that the nth term a_n approaches some real number as n increases. If, for example,

$$a_n = 1 - \frac{1}{n},$$

then a_n is clearly near 1 for large n, A fact we indicate by writing

$$a_n \to 1 \text{ as } n \to \infty \text{ or } \lim_{n\to\infty} a_n = 1$$

or, in words, we say that the sequence $\{1 - 1/n\}$ has limit 1. More formally, we have the following definition.

Definition

A sequence $\{a_n\}$ has the number L as a *limit*, written

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty, \tag{2}$$

if, for every positive number ϵ, there exists a positive integer N such that

$$|a_n - L| < \epsilon \text{ if } n \geq N. \tag{3}$$

Whenever a sequence has a limit, we say that the sequence is *convergent* and that it *converges to the limit L*. A sequence that fails to have a limit is termed *divergent*.

You should think of the above definition as a technical way of saying that as n increases the terms a_n get close and *stay close* to some number L. One way to obtain a visual image is to associate the numbers a_n and L with points on a coordinate line. According to the definition of limit, we may choose any positive number ϵ we please -1, $1/2$, or 10^{-10}, it doesn't matter – and be certain that there is some index N such that each term a_n with index $n \geq N$ lies between $L - \epsilon$ and $L + \epsilon$. In effect, then, once a segment is drawn with center at L, only a finite number of the terms are outside that segment. From this visualization, that the terms of a sequence with limit L must cluster about L, it is clear that a sequence cannot have two limits.

Example 1. Show that $\lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$.

Solution. Here $a_n = \dfrac{n+1}{n}$, $L = 1$ and $a_n - L = a_n - 1 = \dfrac{n+1}{n} - 1 = \frac{1}{n}$.
At the intuitive level, it is clear that $a_n \to 1$ as $n \to \infty$ because $a_n - 1 = \frac{1}{n} \to 0$ as $n \to \infty$. More technically, we must follow the procedures set forth in the definition of a limit. Let ϵ be a positive number and pose the following question: How large must we choose n in order that

$$|a_n - 1| = \frac{1}{n} < \epsilon?$$

Obviously, $\frac{1}{n} < \epsilon$ if $\frac{1}{\epsilon} < n$. So if we let N be the smallest integer such that $\frac{1}{\epsilon} < N$, then $n \geq N$ implies that $n > \frac{1}{\epsilon}$ and, hence, that

$$|a_n - 1| < \epsilon.$$

This analysis amounts to a strategy for finding N once ϵ is chosen. For example, if $\epsilon = 0.5$, $\frac{1}{\epsilon} = 2$ and we choose $N = 3$; if $\epsilon = 0.025$, $\frac{1}{\epsilon} = 40$ and $N = 41$; and if $\epsilon = 0.00001$, $\frac{1}{\epsilon} = 100,000$ and $N = 100,001$. Two things are obvious: first, the choice of N depends on the choice of ϵ and, second, making ϵ small forces N to be large. ‡

Example 2. Show that the sequence a_n with $a_n = (-1)^n(1 + 1/n)$ is divergent.

Solution. Since $(-1)^n$ is -1 if n is odd and 1 is odd and 1 if n is even, the terms a_n oscillate from slightly larger than 1 for even n to slightly less than
-1 for odd n. The terms of this sequence clearly can't get close and stay close to any number. In fact any two consecutive terms differ by more than 2 in absolute value. ‡

Example 3. Show that $\lim\limits_{n\to\infty} \dfrac{3n^2 - n + 1}{2n^2 + 5n} = 3/2$.

Solution. The nth term of this sequence is $a_n = \dfrac{3n^2 - n + 1}{2n^2 + 5n}$ and at the intuitive level it is fairly clear that the limit is 3/2. For if we divide both numerator and denominator by n^2, we have

$$a_n = \frac{3 - 1/n + 1/n^2}{2 + 5/n}$$

and since $3 - 1/n + 1/n^2 \to 3$ and $2 + 5/n \to 2$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 - 1/n + 1/n^2}{2 + 5/n}$$

and since $3 - 1/n + 1/n^2 \to 3$ and $2 + 5/n \to 2$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 - 1/n + 1/n^2}{2 + 5/n} = \frac{3}{2}.$$

To follow a more rigorous approach, we examine the difference

$$a_n - \frac{3}{2} = \frac{3n^2 - n + 1}{2n^2 + 5n} - \frac{3}{2} = \frac{-17n + 2}{2(2n^2 + 5n)}$$

and show that the expression on the right tends to 0 as $n \to \infty$. First, if $2 \le n$,

$$\left|a_n - \frac{3}{2}\right| = \left|\frac{-17n + 2}{|4n^2 + 10n|}\right| = \frac{|-17n + 2|}{|4n^2 + 10n|} \le \frac{|-17n| + 2}{|4n^2 + 10n|}$$

$$= \frac{17n + 2}{4n^2 + 10n} \le \frac{17n + n}{4n^2 + 0} = \frac{18n}{4n^2} = \frac{9}{2n}.$$

Given a positive number ϵ, we want to be certain than $9/2n < \epsilon$ for n sufficiently large. Following the technical of Example 1, we choose N as the smallest integer such that $9/2\epsilon < N$. Then, if $n \ge N$ if follows that $9/2n \le 9/2N \le 9/2N < \epsilon$ and hence that

$$|a_n - \frac{3}{2}| < \epsilon.$$

Again, we have a rule that computes N in terms of ϵ. For example, if $\epsilon = 0.0017$, $9/2\epsilon = 2647.05...$ and $N = 2648$. ‡

This example illustrates both the intuitive approach and a strict application of the limit definition. In practice, we shall not insist on a strict or rigorous approach, rather preferring to attack limit problems with a blend of common sense and calculus.

If f is a function whose domain includes the integer k and all $x \ge k$, then we can define a sequence by the formula

$$a_n = f(n),\ \ n = k, k+1, k+2, \ldots .$$

and if it turns out that f has a horizontal asymptote,

$$\lim_{n\to\infty} f(x) = L,$$

then the sequence must do the same, that is, we must also have $\lim_{n\to\infty} a_n = L$. Thus, from the function $f(x) = (1+2/x)^x$ comes the sequence $a_n = (1+2/n)^n$. This correspondence together with our previous work on indeterminate forms provides a useful method for finding limits.

Example 4. Find $\lim_{n\to\infty} \frac{\ln\ n}{n}$.

Solution. The sequence $a_n = \frac{\ln\ n}{n}$ is associated with $f(x) = \frac{\ln\ x}{x}$, a function that assumes the indeterminate form ∞/∞ as $x \to \infty$. Hence we may apply L'Hospital's rule to obtain

$$\lim_{n\to\infty} \frac{l^x}{x} = \lim_{n\to\infty} \frac{1/x}{1} = \lim_{n\to\infty} \frac{1}{x} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{\ln\ n}{n} = 0. \qquad \ddagger$$

As we saw in Example 2, a sequence may diverge because its terms oscillate, never clustering around a fixed value. A sequence also may diverge because its terms increase or decrease without bound. This is certainly the case for the sequences

$$1^2, 2^2, 3^2, ..., n^2, ...,$$

$$\ln\ 1,\ \ln\ 2,\ \ln\ 3, ..., \ln\ n, ...\ ;$$

situations which we describe by writing

$$\lim_{n\to\infty} n^2 = \infty,\ \lim_{n\to\infty} \ln\ n = \infty,\ \lim_{n\to\infty} -\sqrt{n} = -\infty$$

In general we write

(4) $$\lim_{n\to\infty} a_n = \infty$$

to mean that given a number M there is an integer N such that

$$a_n > M \text{ for } n \geq N.$$

A similar meaning is attached to

(5) $$\lim_{n\to\infty} a_n = -\infty.$$

Such sequences are divergent; they have no limit. Nonetheless, we describe (4) by saying that $\{a_n\}$ diverges to ∞; the sequence in (5) diverges to $-\infty$. It is worth noting that the reciprocal of a sequence that diverges to ∞ or $-\infty$ is a convergent sequence with limit 0. Thus,

$$\lim_{n\to\infty} \ln\ n = \infty \text{ but } \lim_{n\to\infty} \frac{1}{\ln\ n} = 0$$

and

$$\lim_{n\to\infty} n^3 = \infty \text{ but } \lim_{n\to\infty} \frac{1}{n^3} = 0.$$

The following theorem catalogues some useful limits.

Theorem 1.

(6) $$\lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1.$$

(7) $$\lim_{n\to\infty} \frac{1}{n^k} = 0 \ (k > 0).$$

(8) $$\lim_{n\to\infty} a^{1/n} = 1 \ (a > 0).$$

(9) $$\lim_{n\to\infty} n^{1/n} = 1.$$

(10) $$\lim_{n\to\infty} \left(1 + \tfrac{x}{n}\right)^n = e^x \text{ (all } x)$$

Proof.

(6) We want to show that $|r^n - 0| = |r^n| = |r|^n$ is small for large n. Since $|r| < 1$, $1/|r| > 1$ so $1/|r| = 1 + h$ for some positive h. An induction argument based on

$$(1+h)^2 = 1 + 2h + h^2 > 1 + 2h,$$

$$(1+h)^3 = (1+h)(1+h)^2 > (1+h)(1+2h)$$

$$= 1 = 3h + 2h^2 > 1 + 3h,$$

$$(1+h)^4 = (1+h)(1+h)^3 > (1+h)(1+3h) = 1 + 4h + 3h^2 > 1 + 4h,$$

shows that $(1+h)^n > 1 + nh$ for every positive integer n. Hence

$$|r|^n = \frac{1}{(1+h)^n} < \frac{1}{1+nh} < \frac{1}{nh}.$$

But h is fixed and $1/n \to 0$ as $n \to \infty$. Therefore $1/nh \to 0$ and, in turn, $|r|^n \to 0$ as $n \to \infty$.

(7) By the definition of exponents, $n^k = e^{k \ln n}$. Now, if $k > 0$, $k \ln n \to \infty$ as $n \to \infty$ and $e^x \to \infty$ as $x \to \infty$. Hence $n^k \to \infty$ as $n \to \infty$ and, therefore, its reciprocal $1/n^k$ must tend to 0.

(8) Let $b_n = a^{1/n}$, $a > 0$. Then $\ln b_n = \frac{1}{n} \ln a \to 0$ as $n \to \infty$. Hence

$$b_n = e^{\ln b_n} \to e^0 = 1 \text{ as } n \to \infty.$$

Here we have used the fact that the function e^x is continuous at $x = 0$.

(9) Let $a_n = n^{1/n}$. Then $\ln a_n = \frac{1}{n} \ln n \to 0$ as $n \to \infty$ by Example 4. Hence

$$a_n = e^{\ln a_n} \to e^0 = 1 \text{ as } n \to f.$$

(10) Fix x and consider $f(t) = \left(1 + \frac{x}{t}\right)^t$ as $t \to \infty$. Since $y = \ln f(t) = t \ln \left(1 + \frac{x}{t}\right)$ has the indeterminate form $\infty \cdot 0$ as $t \to \infty$, we may apply L'Hospital's rule to see that

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{\ln\left(1 + \frac{x}{t}\right)}{\frac{1}{t}} = \lim_{t\to\infty} \frac{\frac{1}{1+\frac{x}{t}} \cdot \frac{-x}{t^2}}{\frac{-1}{t^2}} = \lim_{t\to\infty} \frac{x}{1 + \frac{x}{t}} = x$$

and hence that $f(t) = e^y \to e^x$ as $t \to \infty$. The result now follows in view of the correspondence $\left(1 + \frac{x}{n}\right)^n = f(n)$. ‡

Sometimes it is possible to determine the limit of a sequence by comparing its terms with the terms of sequences having known limits.

Theorem 2. *Pinching Principle for Sequences*

If the sequences $\{a_n\}$ and $\{b_n\}$ have a common limit L and $\{b_n\}$ is a sequence such that

$$a_n \leq b_n \leq c_n, \tag{11}$$

then $\{b_n\}$ also has limit L.

Proof. Choose a positive number ϵ and draw an open interval about L running from $L-\epsilon$ to $L+\epsilon$. Since both $\{a_n\}$ and $\{c_n\}$ converge to L, it is possible to find an N such that a_n and c_n belong to $(L-\epsilon,\ L+\epsilon)$ for $n \geq N$. But b_n is *between* a_n and c_n. Therefore,

$$|b_n - L| < \epsilon \text{ for } n \geq N.$$

This proves that $\lim\limits_{n\to\infty} b_n = L$. ‡

This result is especially useful in the case $a_n = 0$ for all n. The assertion then amounts to saying that

$$\text{if } 0 \leq b_n \leq c_n \tag{12}$$

and $\lim\limits_{n\to\infty} c_n = 0$, then $\lim\limits_{n\to\infty} b_n = 0$.

Computations with limits is sometimes aided by the application of some almost self-evident rules. We omit the formal proofs.

Theorem 3. *If $\{a_n\}$ has limit A and $\{b_n\}$ has limit B, then*

$$\lim_{n\to\infty} (a_n + b_n) = A + B \tag{13}$$

$$\lim_{n\to\infty} a_n b_n = AB, \tag{14}$$

$$\lim_{n\to\infty} ka_n = kA \quad (k = \text{constant}), \tag{15}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \tfrac{A}{B} \text{ if } B \neq 0,\ b_n \neq 0. \tag{16}$$

A straightforward application of (13), (15), and (16) is given by

$$\lim_{n\to\infty} \frac{3n^2 - n + 1}{2n^2 + 5} = \lim_{n\to\infty} \frac{3 - 1/n + 1/n^2}{2 + 5/n^2} = \frac{\lim\limits_{n\to\infty} (3 - 1/n + 1/n^2}{\lim\limits_{n\to\infty} (2 + 5/n^2}$$

$$= \frac{\lim\limits_{n\to\infty} 3 - \lim\limits_{n\to\infty} (1/n) + \lim\limits_{n\to\infty} (1/n^2)}{\lim\limits_{n\to\infty} 2 + 5 \lim\limits_{n\to\infty} 1/n^2} = \frac{3 - 0 + 0}{2 + 5.0} = 3/2.$$

Theorem 4. If $\lim\limits_{n\to\infty} a_n = L$ and f is a continuous function, then

$$\lim_{n\to\infty} f(a_n) = f(L). \tag{17}$$

Thus the composition of a continuous function f and a convergent sequence $\{a_n\}$ is a new sequence $\{f(a_n)\}$ whose limit is given by (17).

Example 5. Find $\lim\limits_{n\to\infty} \cos\left(\dfrac{2n-3}{3n+4}\right)$.

Solution. Since cos x is continuous and $\lim_{n\to\infty} \frac{2n-3}{3n+4} = \frac{2}{3}$, it follows from (17) that

$$\lim_{n\to\infty} \cos\left(\frac{2n-3}{3n+4}\right) = \cos\left(\lim_{n\to\infty}\frac{2n-3}{3n+4}\right) = \cos\left(\frac{2}{3}\right). \qquad \ddagger$$

2. Monotonic and Bounded Sequences

In the preceding section sequences were shown to be convergent by actually finding the limit. A far more common occurrence is that a given sequence converges, that is, it has a limit, but it is quite impossible to exhibit that limit. Hence it is important to develop methods that guarantee the existence of a limit without actually locating it. For the sequences of concern to us in this section – the monotonic sequences – there is a simple convergence test.

Definition. Monotonic Sequence
A sequence $\{a_n\}$ is
increasing if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$;
strictly increasing if $a_1 < a_2 < a_3 < \cdots < \cdots < a_n < a_{n+1} < \cdots$;
decreasing if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1}$
strictly decreasing if $a_1 > a_2 > a_3 > \cdots > \cdots > a_n > a_{n+1} > \cdots$.

A sequence is termed *monotonic* if any one of these conditions hold.

Technically speaking, the monotonicity of a sequence cannot be determined except by examining the formula for the nth term. Nonetheless, a possible permanent trend can often be detected by examining the first few terms.

Example 1.
(a) sequence: $0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}, ...$ type: strictly increasing
(b) sequence: $1,\ 1,\ \frac{3}{2},\ \frac{3}{2},\ 2,\ 2,\ \frac{5}{4},\ \frac{5}{4}, ...$ type: increasing
(c) sequence: $\frac{2^2}{2!},\ \frac{2^3}{3!},\ \frac{2^4}{4!},\ \frac{2^5}{5!},\ \frac{2^6}{6!}, ...$ type: strictly decreasing
(d) sequence: $0,\ -1,\ -1,\ -2,\ -2,\ -3,\ -3,\ -4, ...$ type: decreasing ‡

Each of the sequences of this example are monotonic, but the sequence

$$1,\ \frac{1}{2},\ 2,\ \frac{1}{3},\ 3,\ \frac{1}{4}, ...$$

is not.

Example 2. Show that the following sequences are monotonic.
(a) $a_n = \ln\ n$
(b) $a_n = \frac{n-1}{n+1}$
(c) $a_n = n^{1/n}$ with domain $n = 3, 4, 5, ...$
(d) $a_{n+1}a_n + 1/a_n,\ a_1 = 1$
(e) $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$
(f) $a_n = \frac{n^2}{2^n}$ with domain $n = 3, 4, 5, ...$

Solutions. (a) $\ln\ n < \ln\ (n+1)$ because $\ln\ x$ has a positive derivative for all positive x.

(b) $a_{n+1} - a_n = \dfrac{n}{n+2} - \dfrac{n-1}{n+1} = \dfrac{2}{(n+2)(n+1)} > 0$ for all n so $\{a_n\}$ is strictly increasing.

(c) Write $\ln\, a_n = \ln\, n^{1/n} = \dfrac{\ln\, n}{n}$. Let $f(x) = \dfrac{\ln\, x}{x}$ and note that $f'(x) = \dfrac{1 - \ln\, x}{x^2} < 0$ if $\ln\, x > 1$ or if $x > e$. Hence $\dfrac{\ln\, x}{x}$ is decreasing if $x > e$. Therefore $\ln\, a_n > \ln\, a_{n+1}$ if $n \geq 3$ or, finally, $a_n > a_{n+1}$ if $n \geq 3$ because $\ln\, x$ increases with x. Consequently the sequence $\{n^{1/n}\}_{n=3}^{\infty}$ is strictly decreasing.

(d) a_1 is positive, and if a_n is positive, then $a_{n+1} = a_n + 1/a_n > 0$. By induction every term is positive. Hence $a_{n+1} = a_n + 1/a_n > a_n$ for all n. The sequence is strictly increasing.

(e) The sequence is strictly increasing since

$$a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{-1}{2(n+1)} + \frac{1}{2n+1} = \frac{1}{2(n+1)(2n+1)} > 0.$$

(f) The sequence is strictly decreasing because

$$\frac{a_n}{a_{n+1}} = \frac{n^2}{2^n} \cdot \frac{2^{n+1}}{(n+1)^2} = \frac{2}{(1+1/n)^2} > 1$$

if $n \geq 3$, that is, $(1+1/n)^2 < 2$ if $n \geq 3$. ‡

The question of convergence for monotonic sequences is especially easy to answer. For example, the strictly increasing sequence $\{n^2\}$ diverges to infinity: its terms do not lie below some fixed value. On the other hand the sequence $\{\frac{n-1}{n}\}$ is strictly increasing with limit 1: its values do lie below some fixed number, in fact, $\frac{n-1}{n} = 1 - \frac{1}{n} < 1$ for all n. The difference is that one sequence is bounded above, the other is not. The following definition formalizes this key observation.

Definition. Bounded Sequence

A sequence $\{a_n\}$ is *bounded above* if there is a constant B such that

$$a_n \leq B \text{ for all } n;$$

it is bound below if there is a constant A such that

$$A \leq a_n \text{ for all } n.$$

A sequence that is both bounded above and below is called a *bounded sequence.*

Not every bounded sequence converges, as one sees by examining the oscillatory sequence $\{(-1)^n\}$, but as the above examples suggest, a bounded and monotonic sequence must converge.

Theorem 5. Every increasing sequence that is bounded above has a limit; and every decreasing sequence that is bounded below has a limit.

This theorem offers no hint as how to find the limit; it merely asserts that under certain conditions the limit will exist. Once it is shown
that $a_n \leq B$ for all n, the theorem tells us that $\lim\limits_{n\to\infty} a_n = L$. The bound B need not be the limit: all we know is that $L \leq B$ and $a_n \leq L$ for all n. If $\{a_n\}$ is strictly increasing, then $a_n < L$ for all n.

We shall regard Theorem 5 as being geometrically obvious. However, it should be noted that its proof rests ultimately on the Least Upper Bound Axiom, an assumption about real numbers that undergirds all of calculus.

LEAST UPPER BOUND AXIOM

> Every set of real numbers that is bounded above has a *least upper bound*; and every set of real numbers that is bounded below has a *greatest lower bound*.

For example, the number 2 is an upper bound for the open interval $0 < x < 1$, so is $\frac{3}{2}$; but 1 is at least upper bound. The limit L of a bound and increasing sequence is the least number which is an upper bound for all its terms.

Example 3. Use Theorem 5 to analyze the convergence of divergence of each sequence in Example 2.

Solution. (a) The sequence $\{\ln n\}$ is divergent; it is increasing but not bounded above.

(b) The sequence $\left\{\frac{n-1}{n+1}\right\}$ is increasing and bounded above by 1, that is, $\frac{n-1}{n+1} \leq 1$ for all n. Hence the sequence has a limit, by Theorem 5.

(c) $\{n^{1/n}\}_{n=3}^{\infty}$ is a decreasing sequence of positive numbers. Hence 0 is a lower bound. Therefore Theorem 5 applies and the sequence has a limit.

(d) The recursive sequence $a_{n+1} = a_n + 1/a_n$, $a_1 = 1$ is increasing and, according to Problem Set 14.1, $a_{n+1} > \sqrt{2n+1}$ for all n. since $\sqrt{2n+1} \to \infty$ as $n \to \infty$, $\{a_n\}$ is not bounded above and $a_n \to \infty$ as $n \to \infty$. There is no limit.

(e) The sequence is increasing, and bounded above because

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} < \frac{n}{n+1} < 1$$

for all n. By Theorem 5, the sequence has a limit.

(f) $\left\{\frac{n^2}{2^n}\right\}_{n=3}^{\infty}$ is a decreasing sequence of positive numbers, hence 0 is a lower bound. Again Theorem 5 tells us that the sequence has a limit. ‡

Whether a sequence $\{a_n\}$ has a limit depends on the behavior of a_n for large n; terms at the beginning have no effect. In practice this means you may ignore any initial block. Put technically, if $\{a_n\}_{n=k}^{\infty}$ has limit L, then $\{a_n\}_{n=1}^{\infty}$ will also have limit L. Thus, in reference to Theorem 5, a sequence that is eventually increasing (increasing from some term on) and bounded above has a limit; a sequence that is eventually decreasing and bound below has a limit.

Example 4. Use Theorem 5 to establish that the sequence $\left\{\frac{n^{10}}{e^n}\right\}_{n=1}^{\infty}$ has a limit.

Solution. The sequence $a_n = n^{10}/e^n$ increases through the tenth term a_{10} and thereafter decreasing, as may be seen by differentiating $f(x) = x^{10}/e^x$. Hence Theorem 5 applies and a limit exists. The limit is 0 as may be seen by successive applications of L'Hospital's rule. ‡

3. Convergent and Divergent Series, Partial Sums

The most familiar finite series is the *finite geometric series*

$$1 + x + x^2 \cdots + x^{n-1} \tag{1}$$

whose sum s_n is given by

$$s_n = \frac{1 - x^n}{1 - x} = \frac{1}{1 - x} - \frac{x^n}{1 - x} \text{ if } x \neq 1. \tag{2}$$

Of the two terms on the right in (2), one involves n, the other does not; and in the second term, if $|x| < 1$, $x^n \to 0$ as $n \to \infty$, by (6) of Theorem 1. Hence

$$s_n \to \frac{1}{1 - x} \text{ as } n \to \infty \text{ if } |x| < 1.$$

Or, saying the same thing,

$$1 + x + x^2 + \cdots + x^{n-1} \to \frac{1}{1 - x} \text{ as } n \to \infty \text{ if } |x| < 1.$$

We describe this limit by writing an infinite series:

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \text{ if } |x| < 1.$$

The three dots are important; they serve to indicate that the established pattern continues without end. Thus, if $x = 1/2$,

$$+\frac{1}{2} + \frac{1}{2^2} + + \cdots = \frac{1}{1 + \frac{1}{2}} = 2,$$

meaning that 2 is the limit of the partial sums

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \tfrac{1}{2} \\ s_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{2^2} \\ &\vdots \\ s_n &= 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{2^n} \\ &\vdots \end{aligned}$$

This example illustrates the basic concept of an infinite series.

Definition. Any sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers can be used to form an *infinite series*

$$a_1 + a_2 + a_3 + \cdots, \tag{3}$$

which is said to be *convergent* if its *sequence of partial sums*

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n \tag{4}$$

converges to a limit S. The limit S is called the *sum* of the infinite series (3) and we write

$$S = a_1 + a_2 + a_3 + \cdots. \tag{5}$$

In case the sequence of partial sums (4) fails to converge, we say that the infinite series (3) is *divergent*.

The numbers a_n are the *terms* of the series (3); a_1 is the first term, a_2 is the second term, etc. They can be computed from the partial sums s_n by the formula

$$a_1 = s_1,\ a_2 = s_2 - s_1,\ a_3 = s_3 - s_2, \cdots .$$

For clarity, we shall often expand the notation in (3) to read

$$a_1 + a_2 + a_3 + \cdots + \cdots ,$$

especially if we wish to remove any doubt as to the formula for a_n. And if brevity dictates, we shall adopt the summation notation

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n.$$

In many problems it is convenient to begin indexing an infinite series at 0 instead of 1. We then write

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=0}^{\infty} a_n.$$

In this setting the partial sums are $s_0 = a_0,\ s_1 = a_0 + a_1$, etc.

Example 1. Find the sum S of the series $1 + \frac{5}{8} + \left(\frac{5}{8}\right)^2 + \left(\frac{5}{8}\right)^3 + \cdots$.

Solution. The partial sums are

$$\begin{aligned}
s_1 &= 1, \\
s_2 &= 1 + \tfrac{5}{8}, \\
&\vdots \\
s_n &= 1 + \tfrac{5}{8} + \left(\tfrac{5}{8}\right)^2 + \cdots + \left(\tfrac{5}{8}\right)^n = \frac{1 - \left(\frac{5}{8}\right)^{n+1}}{1 - \frac{5}{8}}
\end{aligned}$$

and, since $\lim_{n\to\infty} \left(\frac{5}{8}\right)^{n+1} = 0$, it follows that $\lim_{n\to\infty} s_n = 8/3$. Hence the series has sum $S = 8/3$, that is,

$$\frac{8}{3} = 1 + \frac{5}{8} + \left(\frac{5}{8}\right)^2 + \left(\frac{5}{8}\right)^3 + \cdots . \ddagger$$

Example 2. Show that the geometric series $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$ diverges if $|x| \geq 1$.

Solution. At $x = 1$ the geometric series reduces to

$$1 + 1 + 1 + \cdots ,$$

a divergent series because its sequence of partial sums

$$s_1 = 1,\ s_2 = 2,\ s_3 = 3, \cdots , s_n = n \cdots$$

diverge to infinity. At $x = -1$, the geometric series reduces to

$$1 - 1 + 1 - 1 + \cdots ,$$

and this series diverges because its partial sums

$$s_1 = 1,\ s_2 = 0,\ s_3 = 1,\ s_4 = 0, \cdots$$

oscillate between 0 and 1 and therefore fail to have a limit.

If $x > 1$ the geometric series diverges because, by (2),

$$s_n = \frac{1-x^n}{1-x} = \frac{x^n = 1}{x-1} \to \infty \text{ as } n \to \infty.$$

And if $x < -1$, the partial sums oscillate wildly with $s_{2n} \to \infty$ and $s_{2n+1} \to -\infty$ as $n \to \infty$. ‡

This example and our opening remarks establish the following important theorem.

Theorem 6. (Geometric Series). The geometric series

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

converges to $\dfrac{1}{1-x}$ if $|x| < 1$; that is,

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \text{ if } |x| < 1; \tag{6}$$

and the geometric series diverges if $|x| \geq 1$.

In (6) the number x is called the *common ratio.*

Example 3. As a consequence of Theorem 6:

(a) $1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 + \cdots = \frac{1}{1-\left(\frac{2}{3}\right)} = \frac{3}{5}$ because $\left|-\frac{2}{3}\right| < 1$;

(b) $1 + e^{-1} + e^{-2} + e^{-3} + \cdots = \frac{1}{1-\frac{1}{3}} = \frac{e}{e-1}$ because $e^{-1} < 1$;

(c) $1 + e + e^2 + e^3 + \cdots$ diverges because $e > 1$. ‡

One of the most illuminating of the many applications of the geometric series is to the study of decimal expansions of rational numbers. The connection arises from the observation that since .9 means $9/10$ and .99 means $9/10 + 9/10^2$, then the only logical meaning for the *infinite decimal* .999... is the infinite series $9/10 + 9/10^2 + 9/10^3 + \cdots + 9/10^n + \cdots$. The sum of this series is, as always, the limit of its partial sums, of which the nth is

$$\begin{aligned} s_n &= \tfrac{9}{10} + \tfrac{9}{10^2} + \tfrac{9}{10^3} + \tfrac{9}{10^n} \\ &= \tfrac{9}{10}\left[1 + \tfrac{1}{10} + \tfrac{1}{10^2} + \cdots + \tfrac{1}{10^{n-1}}\right] \\ &= \tfrac{9}{10} = \tfrac{1-1/10^n}{1-1/10} = 1 - 1/10^n. \end{aligned}$$

Since $\lim\limits_{n\to\infty} s_n = 1$, we conclude that

$$.999... = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots = 1.$$

In general, if $\{a_n\}_{n=0}^{\infty}$ is a sequence of integers taken from 0 through 9, the infinite decimal $.a_1, a_2a_3 \cdots$ denotes the infinite series

$$\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{a_1}{10} + \frac{a_2}{10^2} + + \frac{a_3}{10^3} + + \cdots + \frac{a_n}{10^n} + \cdots$$

And if the a_n are periodic, in the sense that a certain pattern is repeated, then the decimal represents a rational number which can be found by a procedure similar to that just illustrated.

Example 4. Find the rational number represented by .444....

Solution. This decimal corresponds to the series

$$\sum_{n=1}^{\infty} \frac{4}{10^n} = \frac{4}{10} + \frac{4}{10^2} + \frac{4}{10^3} + \cdots$$

which has as its partial sum

$$\begin{aligned} s_n &= \tfrac{4}{10} + \tfrac{4}{10^2} + \cdots \tfrac{4}{10^n} \\ &= \tfrac{4}{10}\left[1 + \tfrac{1}{10} + \cdots + \tfrac{1}{10^{n-1}}\right] = \tfrac{4}{10}\tfrac{1-1/10^n}{1-1/10} = \tfrac{4}{9}(1 - 1/10^n). \end{aligned}$$

Since $\lim_{n\to\infty} s_n = 4/9$, it follows that $.444... = 4/9$. ‡

You should have observed by now that the terms of a convergent series are ultimately very small.

Theorem 7. If the series $\sum_{k=1}^{\infty} a_k$ tend to a finite limit as $n \to \infty$. Therefore, $a_n = s_{n+1} - s_n \to S - S = 0$ as $n \to \infty$. ‡

The content of this theorem is of more immediate use when stated as a

SUFFICIENT CONDITION FOR DIVERGENCE

(7) If the sequence $\{a_n\}$ either fails to have a limit or has a limit different from 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 5. In consequence of (7), the following series are divergent.

(a) $$\sum_{n=1}^{\infty} \tfrac{n}{n+1} = \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \cdots, \; a_n \to 1 \text{ as } n \to \infty$$

(b) $$\sum_{n=1}^{\infty} \tfrac{(-1)^n n}{n+1} = -\tfrac{1}{2} + \tfrac{2}{3} - \tfrac{3}{4} + \cdots, \; a_n \to 1 \text{ as } n \to \infty$$

(c) $$\sum_{k=1}^{\infty} \ln\, k = \ln\, 1 + \ln\, 2 + \ln\, 3 + \cdots, a_n \to \infty \text{ as } n \to \infty$$

(d) $$\sum_{k=1}^{\infty} \frac{k^2+k+1}{k^2-k+1} = \tfrac{3}{1} + \tfrac{7}{3} + \tfrac{13}{7} + \cdots, a_n \to 1 \text{ as } n \to \infty$$

(e) $$\begin{aligned} \sum_{j=1}^{\infty} \cos\,(j\pi) &= \cos\,\pi + \cos\,2\pi + \cos\,3\pi + \cdots \\ &= -1 + 1 - 1 + \cdots, a_n \text{ has no limit } \ddagger \end{aligned}$$

Even if $a_n \to 0$ as $n \to \infty$, the series $\sum_{n=1}^{\infty} a_n$ may diverge. There, $a_n = \ln(1 + 1/n) \to 0$ as $n \to \infty$. Yet the series diverges. Such examples are easy to construct. Consider, for example, the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots.$$

The nth term $a_n = \frac{1}{\sqrt{n}}$ certainly tends to 0 as $n \to \infty$. Nonetheless, the series diverges because

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \frac{n}{\sqrt{n}} = \sqrt{n} \to \infty \text{ as } n \to \infty.$$

The identical phenomenon is present in the *harmonic series*

$$\sum_{n=1}^{\infty} \tfrac{1}{n} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots \tag{8}$$

whose partial sums grow ever so slowly; but they do grow. One must add 12,367 terms to obtain a partial sum that exceeds 10, and 33,617 terms are needed to exceed 11. At this latter stage, where each additional term adds less than 3×10^{-5}, it seems far more likely that the partial sums will level off and approach a limit. That they continue to grow may be seen by clever grouping:

$$\begin{aligned}
s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= \left(1 + \tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \tfrac{1}{2} \\
s_8 &= \left(1 + \tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} \\
s_{2^m} &= \left(1 + \tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{2^{m-1}+1} + \cdots + \tfrac{1}{2^m}\right) > 1 + \tfrac{m}{2}.
\end{aligned}$$

This means that the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, s_{64}, \ldots$ diverge to infinity. However, $s_1 < s_2 < s_3 < s_4 < \cdots$, so we conclude that the entire sequence of partial sums diverges to infinity and the series diverges. This property of the harmonic series merits a special statement.

Theorem 8. The harmonic series diverges.

The simple rules for adding two convergent series and multiplying a series by a constant are set forth in the following theorem. The more complicated rules for multiplying and dividing infinite series will be touched on later.

Theorem 9. Suppose $\sum_{n=1}^{\infty} (a_n + b_n)$ is a convergent series and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B. \tag{9}$$

Also $\sum_{n=1}^{\infty} c a_n$ is convergent for every constant c and

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n = cA. \tag{10}$$

Proof. By assumption $s_n = \sum_{n=1}^{\infty} a_k \to A$ and $t_n = \sum_{k=1}^{\infty} b_k \to B$ as $n \to \infty$, and $s_n + t_n = \sum_{k=1}^{\infty} (a_k + b_k)$ is the nth partial sum for $\sum_{k=1}^{\infty} (a_k + b_k)$. The limit of these partial sums is $\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = A + B$, by Theorem 3. This proves (9). For (10), write $p_n = \sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k = cs_n$ so that $\lim_{n\to\infty} cs_n = c\lim_{n\to\infty} s_n = cA$, again by Theorem 3. This proves (10). ‡

Notice that rules (9) and (10) resemble the two basic rules of integration

$$\int_a^b (f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

and

$$\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx.$$

In practice, they are applied in an identical manner. For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}.$$

By (9),

$$\begin{aligned} \tfrac{7}{4} = 1 + \tfrac{3}{4} &= \sum_{n=1}^{\infty} \tfrac{1}{n(n+1)} + \sum_{n=1}^{\infty} \tfrac{1}{n(n+2)} \\ &= \sum_{n=1}^{\infty} \left[\tfrac{1}{(n+1)} + \tfrac{1}{(n+2)}\right] = \sum_{n=1}^{\infty} \tfrac{2n+3}{n(n+1)(n+2)} \end{aligned}$$

Also,

$$\begin{aligned} \tfrac{1}{4} = 1 - \tfrac{3}{4} &= \sum_{n=1}^{\infty} - \tfrac{1}{n(n+1)} - \sum_{n=1}^{\infty} \tfrac{1}{n(n+2)} \\ &= \sum_{n=1}^{\infty} \left[\tfrac{1}{n(n+1)} - \sum_{n=1}^{\infty} \tfrac{1}{n(n+2)}\right] = \sum_{n=1}^{\infty} \tfrac{1}{n(n+1)(n+2)}. \end{aligned}$$

This obvious rule for subtracting two convergent series results from a combination of (9) and (10), using the choice $c = -1$:

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} (a_n + (-b_n)) &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (-b_n) \\ &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = A - B. \end{aligned}$$

Example 6. Find the sum of the series

$$\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots.$$

Solution. This is a geometric series $a + ax + ax^2 + \cdots$ in which $a = \left(\frac{1}{2}\right)^3$. We simply factor a out:

$$\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^5 + \cdots + = \left(\tfrac{1}{2}\right)^3 \left[1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots\right]$$

$$= \left(\tfrac{1}{2}\right)^3 \tfrac{1}{1-1/2} = \tfrac{1}{4}. \ddagger$$

Rule (10) applies only to convergent series. But there is a related idea which is often useful in deciding a series diverges. It is this:

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ diverges and } c \neq 0, \text{ then } \sum_{n=1}^{\infty} c a_n \text{ diverges.} \tag{11}$$

Thus a known divergent series can't be turned into a convergent series by multiplying each term by a constant, unless that constant is zero. For each partial sum of the second series is the product of c and the corresponding partial sum of the first series; therefore if one diverges so must the other.

Example 7.

(a) $$\sum_{n=1}^{\infty} \tfrac{5}{n} \text{ diverges because } \sum_{n=1}^{\infty} \tfrac{1}{n} \text{ diverges.}$$

(b) $$\sum_{n=1}^{\infty} \tfrac{3n}{n+1} \text{ diverges because } \sum_{n=1}^{\infty} \tfrac{n}{n+1} \text{ diverges. } \ddagger$$

To this point we have used only series starting with a_0 or a_1, but it is also sometimes necessary to use series that begin with a_k:

$$\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots.$$

Indeed, any convergent series can be written as the sum of an initial block of terms plus another convergent series. Thus,

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n,$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n,$$

and, in general,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_k + \sum_{n=k+1}^{\infty} a_n \tag{12}$$

for each positive integer k. This formula, an easy consequence of comparing partial sums of the two series, makes no sense for divergent series.

Example 8.

(a) $$\sum_{n=1}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n \text{ if } |x| < 1.$$

(b) $$\sum_{n=1}^{\infty} \tfrac{n}{n(n+1)} = \tfrac{1}{2} + \tfrac{1}{6} + \sum_{n=3}^{\infty} \tfrac{1}{n(n+1)}.$$

Example 9. Find the sum of the series $\sum_{k=3}^{\infty} 1/2^k$.

By (12),

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2^2} + \sum_{k=3}^{\infty} \frac{1}{2^k};$$

also

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1 - 1/2} = 2.$$

Therefore,

$$\begin{aligned} \sum_{k=3}^{\infty} \tfrac{1}{2^n} &= \sum_{k=0}^{\infty} \tfrac{1}{2^k} - 1\tfrac{1}{2} - \tfrac{1}{2^2} \\ &= 2 - \tfrac{7}{4} = \tfrac{1}{4}. \end{aligned}$$

‡

Whether a series converges or diverges has nothing to do with where the indexing starts. This simple observation saves considerable fussing in practice. Specifically, if the series $\sum_{n=k}^{\infty} a_n$ converges for some starting index k, it will converge for any other starting index; and if $\sum_{n=k}^{\infty} a_n$ diverges for some starting index, it will diverge for any other starting index. To illustrate, since

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$$

is a convergent series, so is the series

$$\sum_{n=100}^{\infty} \frac{1}{2^n} = \frac{1}{2^{100}} + \frac{1}{2^{101}} + \frac{1}{2^{102}} + \cdots;$$

and since the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges, the series

$$\sum_{n=1}^{\infty} \frac{1}{n+5} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

is also divergent.

4. Series of Positive Terms

More often than not the question of convergence for an infinite series cannot be settled with the ease suggested by the preceding section, the difficulty being that it is not usually possible to find a simple formula for the nth partial sums $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ of the series $\sum a_n$. Consequently, the technique of establishing convergence by directly finding the limit of the sequence s_n must be abandoned in favor of various indirect approaches. The most basic of these methods of *testing* a series for convergence apply to series in which all the terms are positive. For brevity we shall refer to a series $\sum a_n$ in which $a_n > 0$ as a *series of positive terms.* For such a series we always have

$$s_n = s_{n-1} + a_n > s_{n-1}$$

so that the partial sums form a strictly increasing sequence. If the partial sums are also bounded, that is, if there is a constant M such that $s_n \leq M$ for all n, then the series must converge because, by Theorem 5, every increasing and bounded sequence has a limit. But if the partial sums are not bounded, then the series diverges. These remarks establish a basic principle about series of positive terms.

Theorem 10. A series $\sum a_n$ of positive terms converges if its partial sums are bounded, and diverges if they are not.

In many series the nth term a_n is the value of a simple function f at n,

$$a_n = f(n)$$

and it becomes fruitful to analyze the partial sums of

$$a_1 + a_2 + a_3 + \cdots = f(1) + f(2) + f(3) + \cdots$$

by comparing areas. The comparison is particularly simple if the associated function f is continuous, positive, and decreasing.

Theorem 11. The Integral Test

Let f be continuous, positive, and decreasing on the interval $x \geq 1$. Then the series

$$f(1) + f(2) + f(3) + \cdots \tag{1}$$

and the improper integral

$$\int_1^\infty f(x)\,dx \tag{2}$$

either both converge or both diverge.

Proof. Since f is decreasing on each of the unit intervals $n \leq x \leq n+1$, the area under the graph from n to $n+1$ is less than that of a rectangle of area $f(n)$ (Figure 14.6) and greater than that of a rectangle of area $f(n+1)$ (Figure 14.5). In particular,

$$f(2) < \int_1^2 f(x)\,dx < f(1),$$

$$f(3) < \int_2^3 f(x)\,dx < f(2),$$

$$f(4) < \int_3^4 f(x)\,dx < f(3),$$

$$f(n+1) < \int_n^{n+1} f(x)\,dx < f(n),$$

Addition yields

$$f(2) + f(3) + \cdots + f(n+1) < \int_1^{n+1} f(x)\,dx < f(1) + f(2) + \cdots + f(n).$$

In terms of the partial sums $s_n = \sum_{k=1}^{n} f(k)$, these two inequalities state that

$$s_{n+1} - f(1) < \int_1^{n+1} f(x)\,dx \text{ and } \int_1^{n+1} f(x)\,dx < s_n \tag{3}$$

These comparisons form the basis of the integral test.

The second inequality shows that if the improper integral (2) diverges, that is, if $\int_1^{n+1} f(x)\,dx \to \infty$ as $n \to \infty$, then $s_n \to \infty$ as $n \to \infty$ and the series diverges. But if the improper integral converges, then $\int_1^{n+1} f(x)\,dx$ increases to a limit L as $n \to \infty$ and

$$s_{n+1} < f(1) + \int_1^{n+1} f(x)\,dx < f(1) + L$$

for all n. Hence $\{s_n\}$, being increasing and bounded, has a limit by Theorem 10. ‡

As a first application of the integral test, consider the harmonic series $\sum 1/n$, a known divergent series. Clearly, the test function is $f(x) = 1/x$ because $f(n) = 1/n$. And since $1/x$ is continuous, positive and decreasing on the range $x > 0$, we conclude from

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln\, x]_1^b = \lim_{b\to\infty} \ln\, b = \infty$$

that the harmonic series diverges. Moreover, the second inequality in (3) provides a rough estimate of the rate at which its partial sums grow:

$$\ln\,(n+1) = \int_1^{n+1} \frac{1}{x}\,dx < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = s_n.$$

It is interesting to compare the estimates given by this inequality with some computed values of s_n for selected n:

Estimates	Computed Values
$s_{83} > 4.4308$	$s_{83} = 5.002068...$
$s_{227} > 5.4293$	$s_{227} = 6.004366...$
$s_{616} > 6.4248$	$s_{616} = 7.001274...$
$s_{1,674} > 7.4235$	$s_{1,674} = 8.000485...$
$s_{4,550} > 8.4231$	$s_{4,550} = 9.000208...$
$s_{12,367} > 9.4228$	$s_{12,367} = 10.000043...$

The harmonic series is a special case of the more general *p-series*

$$\sum_{n=1}^{\infty} \tfrac{1}{n^p} = 1 + \tfrac{1}{2^p} + \tfrac{1}{3^p} + \cdots . \tag{4}$$

Example 1. Show that the p-series converges if $p > 1$ and diverges if $p \leq 1$.

Solution. Here the test function is $f(x) = 1/x^p$ because $f(n) = 1/n^p$. It is continuous, positive, and since $f'(x) = -p/x^{p+1} < 0$, it is decreasing on the interval $x > 0$. If $p = 1$, the series (4) is the divergent harmonic series. If $p \neq 1$, the p-series converges or diverges with the improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1 b\, x^{-p}\,dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b = \lim_{b\to\infty}\left(\frac{b^{-p+1}}{1-p} - \frac{1}{1-p}\right).$$

Whether this limit is finite or infinite depends on the exponent $-p+1$. If $-p+1 > 0$ or $p < 1$, b^{-p+1} will increase without bound as b increases; but if $-p+1 < 0$, the exponent on b is negative and $b^{-p+1} \to 0$ as $b \to \infty$. Hence

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \text{ if } p > 1 \text{ and } \int_1^\infty \frac{1}{x^p}\,dx \text{ diverges if } p < 1.$$

According to the integral test this means that the p-series (4) converges if $p > 1$ and diverges if $0 < p < 1$. ‡

In particular, this means that

$$\sum_{n=1}^\infty \frac{1}{n^2} \text{ and } \sum_{n=1}^\infty \frac{1}{n^4}$$

are convergent series ($p = 2$ and $p = 4$) while

$$\sum_{n=1}^\infty \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^\infty \frac{1}{n^{1/3}}$$

are divergent series ($p = 1/2$ and $p = 1/3$). the conclusions appear reasonable because, for instance, $1/n^2$ certainly goes to 0 much faster than $1/\sqrt{n}$. But, intuitively, no meaningful distinction is apparent between the series

$$\sum_{n=1}^\infty \frac{1}{n^{1.001}} \text{and } \sum_{n=1}^\infty \frac{1}{n^{.999}}.$$

The first converges, the second diverges; yet, for large n their nth terms would seem to be going to zero at about the same rate. But there is a difference, and that difference is enough to force radically different behavior on the respective partial sums.

If a series is shown to converge by the integral test, the sum of the series is *never exactly* the value of the integral. In fact, if $a_n = f(n)$ and f satisfies the conditions of the integral test, then, by (3), the best we can say is that

$$\int_1^\infty f(x)\,dx < \textstyle\sum_{n=1}^\infty a_n < a_1 + \displaystyle\int_1^\infty f(x)\,dx. \tag{5}$$

Hence

$$1 = \int_1^\infty \frac{1}{x^2} < \sum_{n=1}^\infty \frac{1}{n^2} < 1 + \int_1^\infty \frac{1}{x^2}\,dx = 1 + 1 = 2$$

and

$$\frac{1}{p-1} = \int_1^\infty \frac{1}{x^p}\,dx < \sum_{n=1}^\infty \frac{1}{n^p} < 1 + \int_1^\infty \frac{1}{x^p}\,dx = 1 + \frac{1}{p-1} = \frac{p}{p-1} \text{ if } p > 1.$$

Used in this fashion the integral test provides an upper and lower bound for the sum of a convergent series, but the estimates it provides aren't very accurate, as you can tell by examining finite 14.5. A more useful approach consists of calculating the sum of the first few terms and using an integral to estimate the tail-end or remainder. Specifically, we write

$$S = \sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \cdots + a_n + \sum_{k=n+1}^\infty a_k$$

and call

$$R_n = \sum_{k=n+1}^\infty a_k \tag{6}$$

the *remainder after* n-terms. In the setting of the integral test, a comparison of areas shows that

$$R_n < \int_n^\infty f(x)\,dx. \tag{7}$$

Example 2. Estimate the sum of $\displaystyle\sum_{n=1}^\infty 1/n^2$ with an error less than or equal to .005.

Solution. Write

$$S = \sum_{n=1}^\infty \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + R_n,$$

where $R_n = \displaystyle\sum_{n=1}^\infty 1/k^2$. Using $f(x) = 1/x^2$ we find that

$$R_n < \int_n^\infty \frac{1}{x^2}\,dx = \frac{1}{n}.$$

Now the idea is to choose n so large that $R_n < .005$ and this certainly occurs when $n = 200$. Since the partial sums increase to S we have

$$\frac{1}{2^2} + \cdots + \frac{1}{200^2} < S < 1 + \frac{1}{2^2} + \cdots + \frac{1}{200^2} + .005$$

and, with the aid of a calculator, find that

$$1.639946... < S < 1.644946...$$

Actually it can be shown by advanced methods that

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.64493406...$$

Example 3. Use the integral test to show that the series $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

Solution. The test function $f(x) = \dfrac{1}{x \ln x}$ is continuous, positive, and decreasing on the range $x \geq 2$, and

$$\begin{aligned}\int_2^{\infty} \frac{dx}{x \ln x} &= \lim_{b\to\infty} \int_2^b \frac{dx}{x \ln x} = \lim_{b\to\infty} \ln(\ln x)\,\Big|_2^b \\ &= \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty.\end{aligned}$$

Hence the series diverges, even though its terms tend to zero faster than the terms of the divergent series $\sum 1/n$. ‡

In applying the integral test in the preceding example we deviated from the integral test recipe as given in Theorem 10 in that everything was shifted to the right by one unit. A quick review of the idea behind the integral test should convince you that nothing is amiss. Indeed, the integral test applies equally well if the beginning index in (1) and the lower limit in (2) are changed from 1 to any positive integer k.

COMPARISON TESTS

The integral test is of no use in showing that the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots \tag{8}$$

converges, even though the terms are positive and decreasing. The difficulty, of course, is that there is no apparent candidate for $f(x)$. Nonetheless, this series can be analyzed by the idea upon which the integral test is based; namely, that of comparing the terms of one series with those of another. One comparison arises from observing that $2! \geq 2,\ 3! > 2^2,\ 4! > 2^3$ and, in general, if $k > 1$ and one knows that if

$$k! > 2^{k-1}, \text{ then } (k+1)! = k!(k+1) > 2^{k-1}(k+1) > 2^k.$$

This inductive argument shows that $n! \geq 2^{n-1}$ for each positive integer (in fact, equality holds only at $n = 1$). Hence $1/n! \leq 1/2^{n-1}$ and

$$s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = \frac{1 - 1/2^n}{1 - 1/2}$$

$$= 2 - \frac{1}{2^{n-1}} < 2$$

for each positive integer n. This last inequality shows that the partial sums of (8) are bounded above by 2 and that, according to Theorem 10, it converges. Later on we will show that the sum in (8) is $e - 1 = 1.718281828...$

The preceding argument shows that the series $\sum_{n=1}^{\infty} 1/n!$ converges because its terms are less than the corresponding terms of the convergent series $\sum_{n=1}^{\infty} 1/2^{n-1}$. In general, given a series $\sum a_n$ of positive terms and inequalities

$$0 < a_n \leq b_n,$$

for all n, we say that the series $\sum b_n$ *dominates* the series $\sum a_n$.

Theorem 12. Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series of positive terms.

Convergence Test. If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ is also convergent.

Divergence Test. If $0 \leq b_n \leq a_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ is also divergent.

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = s_n = \sum_{k=1}^{n} b_k$. For the Convergence Test, $s_n \leq t_n$ for all n and the partial sums t_n increase to the sum $T = \sum b_n$. Hence $s_n \leq t_n \leq T$ and the partial sums $\{s_n\}$ form an increasing and bounded sequence, that is,

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq \cdots \leq T.$$

The convergence of $\sum a_n$ is guaranteed by Theorem 8.

In the Divergence Test, the relation between s_n and t_n is reversed. Now, $t_n \leq s_n$ because $b_n \leq a_n$ for all n and it is clear that s_n must grow large if t_n does. But t_n tends to infinity because $\sum b_n$ is divergent; hence $s_n \to \infty$ as $n \to \infty$ and $\sum a_n$ diverges. ‡

Example 4. Use the comparison test to determine whether the series $\sum_{n=1}^{n} \frac{\sqrt{n}}{n^2 + 1}$ converges.

Solution. The series is convergent because

$$0 \leq \frac{\sqrt{n}}{n^2 + 1} = \frac{1}{n^{3/2} + 1/\sqrt{n}} < \frac{1}{n^{3/2}}$$

and $\sum_{n=1}^{n} 1/n^{3/2}$ is a known convergent series – the p-series for $p = 3/2$. ‡

Example 5. Apply the comparison test to the series $\sum_{n=1}^{n} \frac{n^{3/2}}{3n^2 - n}$.

Solution. The key observation is that the nth term of this series behaves like $1/3\sqrt{n}$ for large n and that $\sum \frac{1}{3\sqrt{n}} = \frac{1}{3} \sum \frac{1}{\sqrt{n}}$ is a divergent series. This suggests we use the comparison test in its divergence form. Specifically, the inequality

$$\frac{n^{3/2}}{3n^2 - n} = \frac{1}{3\sqrt{n} - \frac{1}{n}} > \frac{1}{3\sqrt{n}} > 0$$

shows that the given series is divergent. ‡

WARNING. In applying the comparison test $0 < a_n \leq b_n$, you can conclude nothing about $\sum a_n$ if the series $\sum b_n$ is divergent. Thus, $0 < 1/n^2 < 1/n$ and $0 < 1/2n < 1/n$; yet $\sum 1/n^2$ converges and $\sum 1/2n$ diverges. Similarly, if $0 < a_n \leq b_n$ and $\sum b_n$ and $\sum a_n$ converges, you can conclude nothing about the series $\sum b_n$.

5. Alternating Series

In this section we turn our attention to series that have positive as well as negative terms. Of these, the simplest are those whose terms are alternating positive and negative. An *alternating* series may be written conveniently in the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

where each $a_n > 0$. The series

$$\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{9}} + \cdots,$$

$$1 - \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots,$$

and

$$1 - 1 + 1 - 1 + \cdots$$

are examples of alternating series. Also, for x in the range $-1 < x < 0$, the geometric series

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \quad (|x| < 1)$$

becomes a convergent alternating series; at $x = -1/2$, it reduces to

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots = \frac{2}{3}.$$

The remarkable feature of an alternating series is that it will converge on the *sole assumption that the a_n decrease to zero.*

Theorem 14. Alternating Series Test

If

(1) $a_n \geq a_{n+1} > 0$ and $a_n \to 0$ as $n \to \infty$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

Moreover, if $S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and s_n is the nth partial sum, then

(2) $$|S - s_n| < a_{n+1}$$

Proof. the idea: partial sums with an odd index decrease toward S while partial sums with an even index increase toward S. Let us first look at the even partial sums – those with even indices. We have

$$s_4 = s_2 + (a_3 - a_4) \geq s_2 \text{ because } a_3 \geq a_4,$$

$$s_6 = s_4 + (a_5 - a_6) \geq s_4 \text{ because } a_5 \geq a_6,$$

and, in general,

$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n} \text{ because } a_{2n+1} \geq a_{2n+2}$$

For the odd partial sums,

$$s_3 = s_1 - a_2 + a_3 = s_1 - (a_2 - a_3) \leq s_1 \text{ because } a_2 \geq a_3,$$

$$a_5 = s_3 - a_4 + a_5 = s_3 - (a_4 - a_5) \leq s_3 \text{ because } a_4 \geq a_5,$$

and, in general,

$$s_{2n+1} = s_{2n-1} = a_{2n} + a_{2n+1} = s_{2n-1} - (a_{2n} - a_{2n+1}) \leq s_{2n-1} \text{ because } a_{2n} \geq a_{2n+1}$$

Further, $s_2 + a_3 = s_3$, $s_4 + a_4 = s_5$, and

(3) $$s_{2n} + a_{2n+1} = s_{2n+1}$$

for all n, so that $s_{2n} < s_{2n+1}$. Hence the situation is exactly as shown in the figure:

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n-2} \leq s_{2n} \leq s_{2n-2} \leq s_{2n} < s_{2n+1} \leq s_{2n-1} \leq \cdots \leq s_5 \leq s_3 \leq s_1.$$

The information we wish to extract from this string of inequalities is that the even partial sums $\{s_{2n}\}$ form an increasing sequence bounded above by s_1 and that the odd partial sums $\{s_{2n+1}\}$ form a decreasing sequence bounded below by s_2. We now call on Theorem 5; it tells us that both sequences converge, one to a number $\underline{S}$, say, the other to $\overline{S}$:

(4) $$\lim_{n\to\infty} s_{2n} = \underline{S}, \ \lim_{n\to\infty} = \overline{S}.$$

To this point, everything we have said depends on the assumptions $a_n \geq a_{n+1} > 0$. The next step uses the assumption $a_n \to 0$ and, in particular, that $a_{2n+1} \to 0$ as $n \to \infty$. From the identities (3) and the limits in (4) it follows that

$$\overline{S} = \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(s_{2n} + a_{2n+1}) = \underline{S} + 0 = \underline{S},$$

that is, $\overline{S} = \underline{S}$. This common value, call it S, is clearly the limit of the entire sequence $\{s_n\}$. We have proved that the partial sums have a limit and hence that the alternating series is convergent.

Proof of inequality (2) rests on the observation that S, the sum of the series, always lies between any two consecutive partial sums; precisely

(5) $$s_{n+1} < S < s_n \text{ if } n \text{ is odd.}$$

and

(6) $$s_n < S < s_{n+1} \text{ if } n \text{ is even.}$$

Since $|s_{n+1} - s_n| = a_{n+1}$, we see that the inequality $|S - s_n| < a_{n+1}$ is merely the statement that the distance from a point inside an interval to one of its endpoints is less than the length of the interval. ‡

The first step in applying the Alternating Series Test is visual: you have to check to be the terms actually alternate in sign. Clearly, the series

(7) $$1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} - \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} - \tfrac{1}{17} - \cdots$$

has positive and negative terms; yet, it is not an alternating series and the Alternating Series Test cannot be applied. Actually, the series (7) diverges. Nor does the alternating series

$$e - e^{1/2} + e^{1/3} - e^{1/4} + \cdots$$

meet the requirements of the Alternating Series Test, even though $a_n = e^{1/n}$ is positive and decrease with n. The difficulty is that $\lim_{n\to\infty} a_n \neq 0$ (see Theorem 1). Thus the series diverges by Theorem 7. But in the series

(8) $$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots,$$

$a_n = 1/n$ is positive, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. The series is therefore convergent because all conditions in (1) are satisfied; accordingly, we say that the series (8) converges by the Alternating Series Test.

Further, according to (2), if S denotes the sum of the series in (8) and we truncate the series after the 199th term, then the partial sum s_{199} approximates S with an error less than $a_{200} = 1/200 = .005$. More accurately, by (5), s_{199} is an overapproximation and s_{200} is an underapproximation to S. Aided by a calculator, we find that

$$.690653... = s_{200} < S < s_{199} = .695653...$$

Naturally, more accurate estimates require more terms; it takes more than a million terms to estimate S to six place accuracy. Later we shall show that

$$S = \ln\, 2 = .693147...$$

Example 1. Show that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1} = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \cdots$$

is convergent.

Solution. In this alternating series $a_n = \frac{n}{n^2+1}$ and the conditions in (1) are most easily verified by examining the continuous function $f(x) = \frac{x}{1+x^2}$. It is clearly positive in the interval $x \geq 1$ and decreasing there since

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$$

if $1 < x$. Because $a_n = f(n)$, it follows that $a_n > a_{n+1} > 0$. And it is clear that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Therefore the series is convergent by the Alternating Series Test. ‡

Example 2. Show that the series

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

is convergent and estimate the error in approximating its sum S by the first nine terms.

Solution. Here $a_n = 1/n!$ and $a_n > a_{n+1} > 0$ because $n!$ increases with n. Also $a_n \to 0$ because $n!$ increases without bound. [Recall that $n! \geq 2^{n-1}$.] Hence the series converges by the Alternating Series Test.

The error in approximating the sum S by

$$s_9 = 1 - \frac{1}{2!} - \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} - \frac{1}{8!} + \frac{1}{9!} = .63212081...$$

is less than

$$a_{10} = \frac{1}{10!} = .000000275...;$$

therefore

$$S \approx .632121,$$

rounded to six decimal places. ‡

Example 3. Show that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} = \frac{3}{2} - \frac{5}{2.3} + \frac{7}{3.4} - \frac{9}{4.5} + \cdots$$

is convergent.

Solution. In this alternating series $a_n = \frac{2n+1}{n(n+1)}$. Clearly, $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$. Rather than proceed as in Example 1, we can establish the inequality $a_n \geq a_{n+1}$ directly by computing the ratio

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{(n+1)(n+2)} \frac{n(n+1)}{2n+1} = \frac{2n^2+3n}{2n^2+5n+2}$$

and then noting the denominator is larger than the numerator if $n \geq 1$. Hence $a_{n+1}/a_n < 1$ or $a_n > a_{n+1}$, and by Theorem 14 the series converges. ‡