1. Absolute Convergence

As we have seen, the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) + \left(-\frac{1}{4}\right) + \cdots \tag{1}$$

is convergent, but the harmonic series

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{3}\right| + \left|-\frac{1}{4}\right| + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

obtained from it by taking the absolute value of each term, is not. Thus tampering with its signs may destroy the convergence of a series – but not necessarily as we see by examining the series

$$1 - \tfrac{1}{2} + \tfrac{1}{2^2} - \tfrac{1}{2^3} + \cdots . \tag{2}$$

This is a convergent geometric series $\sum_{n=0}^{\infty} x^n$ with $x = -1/2$ and the series of absolute values

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{2^2}\right| + \left|-\frac{1}{2^3}\right| + \cdots = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

is also a convergent geometric series. The point is that some convergent series having both positive and negative terms persist in their convergence after each term is replaced by its absolute value – others do not. The underlying concept is called *absolute convergence*.

Definition. Absolute Convergence

The series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is said to be *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots + |a_n| + \cdots \tag{3}$$

is convergent.

According to his definition and the preceding discussion, the series (2) is absolutely convergent; the series (1) is not. Of course, if $a_n \geq 0$ for all n, then

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots = a_1 + a_2 + a_3 + \cdots .$$

Hence a convergent series of positive terms is always absolutely convergent.

Example 1. The series

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots + \frac{(-1)^{n+1}}{(2n-1)^2} + \cdots \tag{a}$$

is absolutely convergent because

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots$$

is convergent. For instance, apply the integral test to $f(x) = \dfrac{1}{(2x-1)^2}$. But the series

(b) $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n+1}}{(2n-1)} + \cdots$$

is not absolute convergent because

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{(2n-1)} + \cdots$$

is a divergent series, as may be seen by applying the integral test to $f(x) = \dfrac{1}{2x-1}$.

It is traditional to give a name to a series which converges but does not converge absolutely. **Definition** The series $\sum a_n$ is called *conditionally convergent* if it is convergent but $\sum |a_n|$ diverges.

This definition simply divides all convergent series into two groups – those that converge absolutely and those that do not. The series

$$\ln\, 2 - \ln\, 3/2 + \ln\, 4/3 - \ln\, 5/4 + \cdots$$

and

$$1 - 1/\sqrt{2} - 1/\sqrt{3} + 1/\sqrt{4} + 1/\sqrt{5} - 1/\sqrt{6} - 1/\sqrt{7} + \cdots$$

are conditionally convergent: each series converges but the corresponding series, with all connecting signs positive, diverge. In such series you should think of convergence as being the result of delicately balanced internal cancellation, a balance that is shattered by making each connecting sign positive. On the other hand, in an absolutely convergent series like

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots,$$

you may change the signs any way you please and the resulting series remains convergent.

We close this section with an important test for absolute convergence.

Theorem 15. The Ratio Test

In the series $\sum a_n$ suppose that $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L$. Then

(a) if $0 \le L < 1$, the series converges absolutely;
(b) if $L > 1$, the series diverges; and
(c) if $L = 1$, the test gives no information concerning convergence.

Proof. We begin with (a). The assertion is that the rations $|\dfrac{a_{n+1}}{a_n}|$ tend to some number L in the interval $0 \le L < 1$. Since the number $r = \dfrac{L+1}{2}$ satisfies the inequality $L < r < 1$, it is possible to choose N large enough so that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \text{ if } n \ge N.$$