Sample project.

Problem:
Given a right triangle with vertices (0,0), (0,a) and (b,0). Inscribe a circle $C_1$, then heading off towards one of the acute angles (say the vertex (b,0)), inscribe an infinite chain of circles $C_n$ each tangent to the preceeding circle and the sides of the angle. Find the total area of the circles.

Solution: Let $r_n$ be the radius of the nth circle. First we get the radius $r_1$ of the 1st circle. Then the center is at $[r_1, r_1]$. Let $(x, y)$ be the point on the hypotenuse where the circles is tangent. Then the slope of the segment from $[r_1, r_1]$ to $(x, y)$ is $\frac{b}{a}$, and the slope of the segment from $(b, 0)$ to $(x, y)$ is $-\frac{a}{b}$. We can use this get x and y in terms of a,b, and r.

```
>  restart;eq1 := (y-r[1])/(x-r[1])=b/a;
```

$$eq1 := \frac{y - r_1}{x - r_1} = \frac{b}{a}$$

```
> eq2 :=  (y-0)/(x-b) = -a/b;
```

$$eq2 := \frac{y}{x - b} = -\frac{a}{b}$$

```
> sol :=solve({eq1,eq2},{x,y});
```

$$sol := \{y = \frac{a\,(a\,r_1 + b^2 - b\,r_1)}{a^2 + b^2}, x = \frac{b\,(-a\,r_1 + b\,r_1 + a^2)}{a^2 + b^2}\}$$

Now the length of the segment from $(x, y)$ to $[r_1, r_1]$ is $r_1$. So, we can get one equation in a,b, and $r_1$ from this which we can solve for $r_1$ in terms of a and b.

```
> eq3 :=  r[1] = simplify(subs(sol,sqrt((x-r[1])^2+(y-r[1])^2)));
```

$$eq3 := r_1 = \sqrt{\frac{(-b\,r_1 + a\,b - a\,r_1)^2}{a^2 + b^2}}$$

Solving for r,

```
> sol2 := solve(eq3,r[1]);
```

$$sol2 := \frac{1}{2}b + \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 + b^2}, \frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}\sqrt{a^2 + b^2}$$

Note: In release 3, sol2 may come up with a RootOf in it. You can get all the roots of this with allvalues. Just uncomment the next line.

```
> # sol2 := allvalues(sol2);
```

By inspection, we can see that r must be the second root. Let's make a function out of this.

```
> rad := unapply(sol2[2],a,b);
```

$$rad := (a, b) \to \frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}\sqrt{a^2 + b^2}$$

Just to check if we're on the right track, let's draw the triangle and the inscribed circle. We make these general so that a and b can be varied.

```
>  tri := (a,b) ->
  plot([[0,0],[b,0],[0,a],[0,0]],scaling=constrained);
```

$$tri := (a, b) \to \mathbf{plot}([[0, 0], [b, 0], [0, a], [0, 0]], scaling = constrained)$$

```
> circle := (c,r) -> plot([c[1]+r*cos(t),c[2]+r*sin(t),t=0..2*Pi]);
```

$$circle := (c, r) \to \mathbf{plot}([c_1 + r\cos(t), c_2 + r\sin(t), t = 0 .. 2\pi])$$

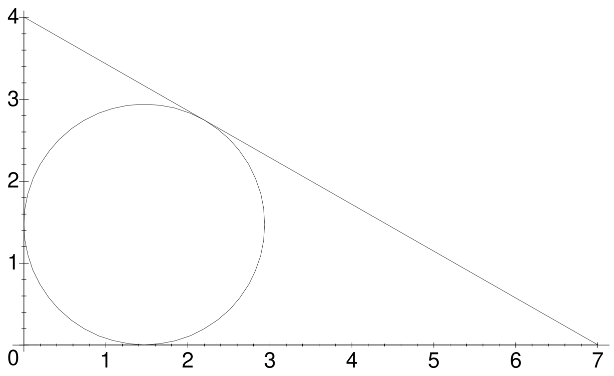
```
> with(plots);
```

[*animate*, *animate3d*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *cylinderplot*, *densityplot*, *display*, *display3d*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *odeplot*, *pareto*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedraplot*, *replot*, *rootlocus*, *semilogplot*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *sphereplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*]

```
>  inscribe := proc(a,b)
     local  r;
     r := rad(a,b);
     plots[display]([tri(a,b),circle([r,r],r)]); end;
```

$$inscribe := \mathbf{proc}(a, b)\ \mathbf{local}\ r;\ r := \mathbf{rad}(a, b);\ plots[display]([\mathbf{tri}(a, b), \mathbf{circle}([r, r], r)])\ \mathbf{end}$$

```
>  inscribe(4,7);
```




Good. This work fine. Now get the ratio of $r_n$ to $r_{n-1}$. Here we can use some trig. Look at the right triangle whose hypotenuse is the segment connecting the centers of the n-1st and nth circles. The hypotenuse is $r_{n-1}+r_n$, the angle α on the lower right vertex is $.5\ \mathbf{arctan}\left(\frac{a}{b}\right)$, and the side opposite this angle is $r_{n-1}+r_n$, which is also $(r_{n-1}+r_n)\sin(\alpha)$.
This is eq4

```
>   eq4 :=  r[n-1]-r[n]= (r[n-1]+r[n])*sin(.5*arctan(a/b));
```

$$eq4 := r_{n-1} - r_n = (r_{n-1}+r_n)\sin\left(.5\arctan\left(\frac{a}{b}\right)\right)$$

Solve this equation for $r_n$

```
> sol := solve(eq4,r[n]);
```

$$sol := -1.\,\frac{r_{n-1}\left(-1. + \sin\left(.5000000000\arctan\left(\frac{a}{b}\right)\right)\right)}{1. + \sin\left(.5000000000\arctan\left(\frac{a}{b}\right)\right)}$$

```
>   collect(sol,r[n-1]) ;
```

$$-1.\,\frac{r_{n-1}\left(-1. + \sin\left(.5000000000\arctan\left(\frac{a}{b}\right)\right)\right)}{1. + \sin\left(.5000000000\arctan\left(\frac{a}{b}\right)\right)}$$

Now we can see what the common ratio is. (Note the lead -1 above. Maple insists on writing things in this way.)

```
> rat := unapply( (1-sin(arctan(a/b)/2))/(1+sin(arctan(a/b)/2))
  ,a,b);
```

$$rat := (a, b) \to \frac{1 - \sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)}{1 + \sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)}$$

Now we can write the total Area of the circles.

```
> Area :=  Sum(Pi*(rad(a,b)*rat(a,b)^i)^2,i=0..infinity);
```

$$Area := \sum_{i=0}^{\infty} \pi \left(\frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}\sqrt{a^2+b^2}\right)^2 \left(\left(\frac{1 - \sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)}{1 + \sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)}\right)^i\right)^2$$

This is a geometric series, which maple can sum. (We could too.)

```
> form := value(Area );
```

$$form := \frac{1}{4}\,\frac{\pi\left(\frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}\sqrt{a^2+b^2}\right)^2\left(1 + \sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)\right)^2}{\sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)}$$

Make a function out of this.

```
> area := unapply(form,a,b);
```

$$area := (a, b) \to \frac{1}{4}\,\frac{\pi\left(\frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}\sqrt{a^2+b^2}\right)^2\left(1 + \sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)\right)^2}{\sin\left(\frac{1}{2}\arctan\left(\frac{a}{b}\right)\right)}$$

```
> area(4.,11.);
```

$$5.386806700\,\pi$$

Now lets draw a picture of the triangle and the first few, say m, circles.

```
> a := 4. : b := 7.:  m := 7: r[1] := rad(a,b);
```

$$r_1 := 1.468871126$$

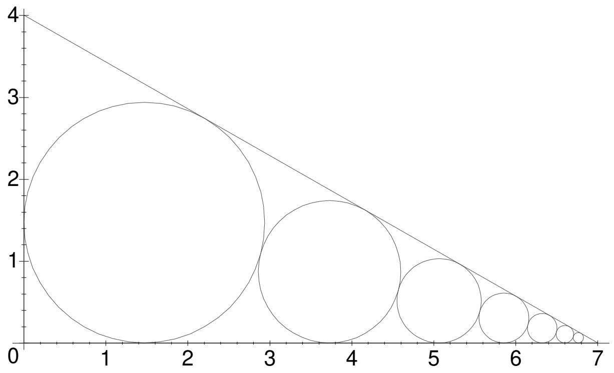
```
> for i from 2 to m do r[i] := rat(a,b)^(i-1)*r[1] od:
> x1 := evalf(r[1]);
```

$$x1 := 1.468871126$$

Get the x coordinate of the center of the ith circle i from 2 to m,
use some trig.

```
> for i from 2 to m do x.i := evalf(x.(i-1) +
  (r[i]+r[i-1])*cos(arctan(a/b)/2)) od:
>  for i from 1 to m do cir.i := circle([x.i,r[i]],r[i]) od:
```

```
> display([tri(a,b),cir1,cir2,cir3,cir4,cir5 ,cir6,cir7]);
```




We can make a word out this.

```
> drawem := proc(a,b,m)
    local i,r,x,cir,ar;
    r[1] := evalf(rad(a,b));
    for i from 2 to m do r[i]:= evalf(rat(a,b)^(i-1)*r[1] )od:
    x[1] := r[1]:
    for i from 2 to m do x[i] :=
               evalf(x[i-1]+ (r[i]+r[i-1])*cos(arctan(a/b)/2)) od:
  for i from 1 to m do cir[i]:= circle([x[i],r[i]],r[i]) od:
     ar := evalf(area(a,b));
     plots[display]([tri(a,b),seq(cir[i],i=1..m)],title=cat(`Area of
  all the circles is`,convert(ar,string)));
     end;
```

***drawem* := proc(*a*, *b*, *m*)**

**local *i*, *r*, *x*, *cir*, *ar*;**

$\boldsymbol{r[1] := \mathrm{evalf}(\mathrm{rad}(a, b));}$

$\boldsymbol{\mathrm{for}\ i\ \mathrm{from}\ 2\ \mathrm{to}\ m\ \mathrm{do}\ r[i] := \mathrm{evalf}(\mathrm{rat}(a, b)\verb|^|(i-1)*r[1])\ \mathrm{od};}$

$\boldsymbol{x[1] := r[1];}$

$\boldsymbol{\mathrm{for}\ i\ \mathrm{from}\ 2\ \mathrm{to}\ m\ \mathrm{do}\ x[i] := \mathrm{evalf}(x[i-1] + (r[i] + r[i-1])*\cos(1/2*\arctan(a/b)))\ \mathrm{od}}$

**;**

$\boldsymbol{\mathrm{for}\ i\ \mathrm{to}\ m\ \mathrm{do}\ cir[i] := \mathrm{circle}([x[i], r[i]], r[i])\ \mathrm{od};}$

***ar*** **:= evalf( area(*a*, *b* ));**

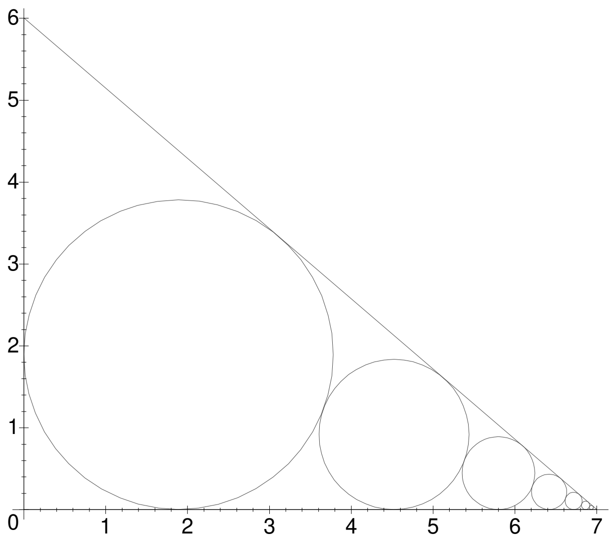
***plots*[*display* ]( [ tri(*a*, *b* ), seq( *cir*[ *i* ], *i* = 1 .. *m* ) ],**

***title* = cat( *'Area of all the circles is'*, convert( *ar*, *string* ) ) )**

**end**

```
> drawem(6,7,7);
```

**Area of all the circles is14.67426964**




```
>
```