MA114 Worksheet for Section 12.1

Skills

1. Verify by substitution that each function is a solution of the given differential equation.
   (a) $y' + 2y = 0; y = 3e^{-2x}$
   (b) $y'' + 4y = 0; y = \cos 2x + \sin 2x$
   (c) $y'' = 9y; y = e^{3x} + e^{-3x}$
   (d) $y'' + 4y' + y = 0; y = xe^{-2x}$
   (e) $y'' - 2y' = 2y = 0; y = e^x(\cos x + \sin x)$
   (f) $y' + 2xy^2 = 0; y = \dfrac{1}{1+x^2}$
   (g) $x^2y'' + xy' - y = \ln x; y = x - \ln x$
2. Show $y(x)$ satisfies the given differential equation, then find values of the constants A and B so that $y(0) = y'(0) = 1$.
   (a) $y'' + 3y' = 0; y(x) = A + Be^{-3x}$
   (b) $y'' - 2y' + y = 0; y(x) = Ae^x + Bxe^x$
3. Determine a solution to the given differential equation by inspection. (That is, guess a solution).
   (a) $y'' = 0$
   (b) $2y' = y$
   (c) $(y')^2 + y^2 = 1$

Concepts

1. A function $y = g(x)$ is described below by some geometric property of its graph. Write a differential equation having the function $g(x)$ as a solution.
   (a) The slope of the graph of g at the point (x, y) is the sum of x and y.
   (b) The tangent line to the graph of g at the point (x, y) intersects the x-axis at the point $(\frac{x}{2}, 0)$.
   (c) Every straight line normal to the graph of g passes through the point $(0, 1)$.
2. Write a differential equation that is a mathematical model of the situation described.
   (a) The rate of change (with respect to time) of a population P is proportional to the square root of the current population.
   (b) The rate of change of the velocity of a coasting motorboat is proportional to the square of the velocity.
   (c) The acceleration of a certain sports car is proportional to the difference between 250 kilometers per hour and the velocity of the car.
   (d) In a city of fixed population P persons, the rate of change with respect time of the number of persons N infected with a certain disease is proportional to the product of the number of people who have the disease and the number of people who do not.
3. A certain environment can support a population of at most M creatures. One possible way to model this is via the equation
   $$\frac{dP}{dt} = kP(M - P),$$
   where $P = P(t)$ is the population at time t and k is a positive constant.
   (a) Write down several physical ("real world") conditions which might restrict population size to a fixed maximum. (For example, a limited food source). Try to think of at least 3 such factors.
   (b) How large is the population when it is growing the fastest?

(c) Under what condition does the population cease to grow?

(d) Suppose $P > M$, that is, the population exceeds what the environment can support. What do you think would happen? Explain how this model accounts for this.

Maple

1. One model for the size $y(t)$ of a population that cannot grow arbitrarily large is the *Gompertz Function*, which is defined to be the solution of $\frac{dy}{dt} = \ln\left(\frac{M}{y}\right) y$, Where M is the maximum size of the population.

   (a) Use Maple's *dsolve* command to solve this differential equation for $y(t)$. (Hint: this will require more than just the *dsolve* command.

   (b) Compute $\lim_{t\to\infty} y(t)$.

   (c) Let $M = 1000$. Find values of the remaining constant so that $y(0) = 500, y(0) = 0, y(0) = 1000$. and $y(0) = 1500$. Plot the solutions coresponding to these initial conditions on the same graph and label each one. Explain the behavior of each curve. Is it realistic?