Chapter 6

Using Maple in Calculus IV

6.1 First order equations

6.1.1 We have seen them before.

The simplest differential equation is the first order equation of the form $y' = f(x)$, where f is some known function of x. We first ran across these in Calculus I. Generally speaking, we try to solve them using `int`. For example, to solve $y' = x^2$, $y = ?$ we would type

```
y := int(x^2,x) + C;
```

$$y := \frac{1}{3}x^3 + C$$

Isaac Newton formalized the concepts of velocity and acceleration. So he probably was the first to solve the second order differential equation $y'' = -32$, $y'(0) = 0$, $y(0) = 100$. $y(t) = ?$

```
yp := int(-32,t);
yp := yp - subs(t=0,yp);
```

$$yp := -32\,t$$

$$yp := -32\,t$$

```
y := int(yp,t);
y := y - subs(t=0,y);
```

Here's a big theorem from Calculus I. Remember it?

Theorem: If two functions have the same derivative on an interval, they differ by a constant on that interval.

This tell us as a corollary that the initial value problem (ivp) $y' = f(x)$, $y(0) = y0$ has a unique solution on any interval containing 0.

We sharpened our ability to solve differential equation of the type $y' = f(x)$ Calculus II by studying the techniques of integration: The main ones are substitution, integration by parts, trig substitution, and partial fractions.

```
y' = 1/x, y(1) = 0,     y = ?
y' = y,  y(0) = 1,      y = ?
y' = sqrt(1-x^2)
y' = 1/(1+x^2)
y' = 1/(1-x^2)
```

We continued in Calculus III with equations of motion:

```
dx/dt = f(t)                Given the velocity vector
dy/dt = g(t)                and one position vector
(x,y) = (x0,y0) at t=0      find all position vectors.

Find (x,y) for all t
```

This is an example of a system of differential equations. So what in general is a differential equation? An equation which expresses a relation between an unknown function y and one or more of its derivatives.

An IVP (Initial Value Problem) is a diffeq together with some initial conditions.

If the relation is a linear relation, then the equation is called a linear equation, otherwise it is nonlinear.

```
y' = f(x) is a linear equation :  so is
y' = y
y' = sin(y) is nonlinear
```

The order of a diffeq is n where n is the highest order derivative appearing in the equation. First order equations: $y' = f(x, y)$ often arise in practical problems: A *solution* of a diffeq is a function which satisfies the equation. There is a word (`dsolve`) in Maple which takes as input a differential equation and possibly some initial values, and attempts to return a solution to the equation. In some sense, you can think of one purpose of this course as finding out how this word works, and learning to set up and solve problems which naturally lead to differential equations. To illustrate how `dsolve` works, use it to solve the above differential equation.

```
restart;  # forget everything.
```

set up the differential equation.

```
diffeq := diff(y(x),x)=x^2;
```

$$diffeq := \frac{\partial}{\partial x} y(\,x\,) = x^2$$

use dsolve on it.

```
dsolve(diffeq,y(x));
```

$$y(\,x\,) = \frac{1}{3}\,x^3 + \_C1$$

Note the constant of integration. If we include the initial conditions, `dsolve` will determine this constant.

```
inits := y(0) = 0;
sol := dsolve({diffeq,inits},y(x));
```

$$inits := y(\,0\,) = 0$$

$$sol := y(\,x\,) = \frac{1}{3}\,x^3$$

Now to plot sol we can just plot the right hand side (rhs) of the solution:

```
plot(rhs(sol),x=-3..3);
```

or we could turn sol into a function with `unapply` and `plot`.

```
f := unapply(rhs(sol),x);
plot(f,-3..3);
```

$$f := x \to \frac{1}{3}\,x^3$$

There is a useful package of words to use called DEtools.

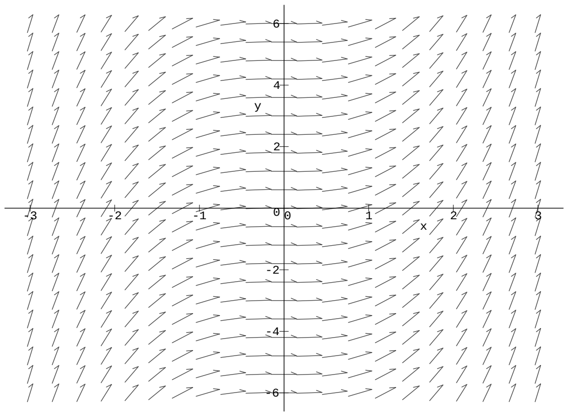
```
with(DEtools);
```

$$[\,DEplot, DEplot1, DEplot2, Dchangevar, PDEplot, dfieldplot, phaseportrait\,]$$

The word `dfieldplot` is one which you can use to examine qualitatively the solution curves to a differential equation y' = f(x,y). For example for the diffeq above,

```
caption := `The direction field plot for y' = x^2.`:
dfieldplot(diffeq,[x,y],x=-3..3,y=-6..6,title=caption);
```

The direction field plot for y' = x^2.



The solutions curves can be seen in the diagram that is drawn. Another term for solution curve is *integral curve*. An *isocline* is a curve that connects points where the slope is constant. For example, the isoclines of the diffeq above are the vertical lines.

One of the important types of equations is the First order linear: `y' + p(x)y = q(x)`. Lots of problems can be solved by setting up and solving a linear differential equation. For example, the salt tank problems give a good place to start.

6.1.2 Salt Tank Problems

a) Make a rough sketch of the amount of salt in the tank over time. Will this function have a horizontal asymptote?

b) Let $q(t)$ be the amount of salt in the tank after t minutes. Express the rate of change of $q(t)$, $q'(t)$, as the rate at which salt is coming into the tank less the rate at which salt is going out of the tank. This will be the differential equation you want to solve.

c) Use `dfieldplot` on your diffeq to get a feel for what the solution curves look like. What are the isoclines?

d) What is q(0)? After you decide, use `dsolve` to solve the initial value problem. Plot your solution and compare with your initial sketch.

e) What is the amount of salt in the tank after 1 minute: One hour? How long does it take the amount of salt in the tank to get to 10 lbs?

6.2 More Salt problems

Let's take a look at that salt tank problem again.

A Salt tank problem

A tank initially contains 50 gallons of fresh water. Brine containing 1/4 lb salt/gal comes in at 3 gals/minute. The mixture drains out at 3 gals/minute. Get the amount of salt in the tank as a function of time.

Solution: Let q be the number of lbs of salt in the tank after t minutes. The rate of change of q with respect to t is the difference of the rate at which salt is coming in which is 3*(1/4) = 3/4 lb per minute and the rate at which salt is going out, which is 3*(q/50).

The initial value problem is `dq/dt = 3(1/4) - 3(q(t)/50), q(0) = 0.`

first let's initialize our variables.

```
t := 't':   the time measured in minutes
q := 'q':   the amount of salt in the tank at time t
```

now define the differential equation.

```
diffeq := diff(q(t),t) = 3*(1/4)- 3*(q(t)/50);
```

$$diffeq := \frac{\partial}{\partial t} q(t) = \frac{3}{4} - \frac{3}{50} q(t)$$

Load the `DEtools` package. (This only needs to be done once in the worksheet.)

```
with(DEtools);
```

$$[DEplot, DEplot1, DEplot2, Dchangevar, PDEplot, dfieldplot, phaseportrait]$$

look at a direction field plot of the equation.

```
dfieldplot(diffeq,[t,q],t=0..60,q=0..15);
```

Note that the isoclines are horizontal lines for this diffeq. Also one of the isoclines is a solution curve. All the other solution curves tend toward this solution curve.

```
solve the equation
sol := dsolve({diffeq,q(0)=0},q(t));
```

$$sol := q(t) = \frac{25}{2}\, e^{(-3/50\,t)}\, e^{(3/50\,t)} - \frac{25}{2}\, e^{(-3/50\,t)}$$

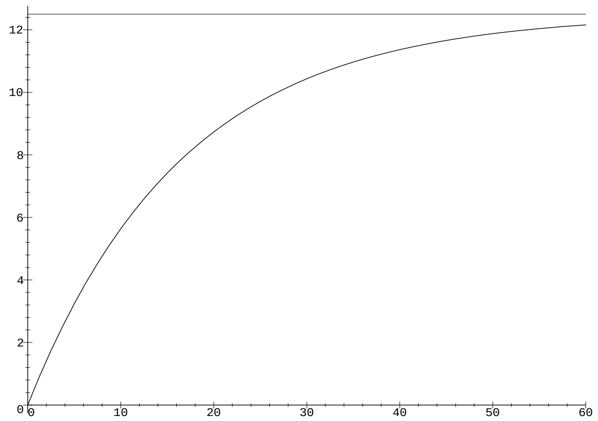
turn the solution into a function.

```
q := unapply(simplify(rhs(sol)),t);
```

$$q := t \to \frac{25}{2} - \frac{25}{2}\, e^{(-3/50\,t)}$$

take a look at the solution over the first hour.

```
plot({q,12.5},0..60);
```



6.2.1 Problems

Problem 1. Answer these questions about the salt tank described above.

a) Use `dfieldplot` on your diffeq to get a feel for what the solution curves to `diffeq` look like. What are the isoclines?

b) What is q(0)? After you decide, use `dsolve` to solve the initial value problem. Plot your solution and compare with your initial sketch.

c) What is the amount of salt in the tank after 1 minute: One hour? How long does it take the amount of salt in the tank to get to 10 lbs?

Problem 2: A second salt tank problem

Now let's alter the problem. Suppose there is a second tank: Initially it contains 50 gallons of brine which contains s lbs of salt. (s is a parameter which we can vary.) Water from the first tank runs into it at 3 gal / minute. let q2 be the amount of salt in the second tank. We can express q2 as a function of t by setting up and solving a differential equation.

```
q2 := `q2`;
diffeq2 := diff(q2(t),t) = 3*(q(t)/50)- 3*(q2(t)/50);
```

$$q2 := q2$$

$$diffeq2 := \frac{\partial}{\partial t}\, q2(\, t\,) = \frac{3}{4} - \frac{3}{4}\, e^{(\,-3/50\,t\,)} - \frac{3}{50}\, q2(\, t\,)$$

```
dsolve({diffeq2,q2(0)=s},q2(t));
```

$$q2(\, t\,) = \frac{25}{2}\, e^{(\,-3/50\,t\,)}\, e^{(3/50\,t\,)} - \frac{3}{4}\, e^{(\,-3/50\,t\,)}\, t + e^{(\,-3/50\,t\,)}\left(-\frac{25}{2} + s\right)$$

Turn q2 into a function of time t with a salt parameter s.

```
q2 := unapply(rhs("),t);
```

$$q2 := t \to \frac{25}{2}\, e^{(\,-3/50\,t\,)}\, e^{(3/50\,t\,)} - \frac{3}{4}\, e^{(\,-3/50\,t\,)}\, t + e^{(\,-3/50\,t\,)}\left(-\frac{25}{2} + s\right)$$

To study the effect of varying s we can use `plots[animate]`, a word from the `plots` package.

```
plots[animate]({q(t),q2(t)},t=0..100,s=0..50);
```

a) What happens to the concentration of the salt in the second tank over time? Make a general observation based on playing with the animation of q2 as s increases from 0 to 50. Is there a limiting concentration? Does the concentration in the second tank ever coincide with the concentration in the first tank? Does the concentration in the second tank ever reach a minimum?

b) Suppose the second tank starts out with 30 lbs of salt. Find when the two tanks have the same amount of salt. Find the minimum amount of salt in tank 2.

c) Consult with your lab partner and come up with two interesting questions related to the salt tank problem. Write them down.

6.3 Money matters

Here's a practical problem. Suppose you borrow A dollars for T years at r percent interest compounded continuously. Assuming you pay this loan back continuously at a constant annual rate of k dollars per year, what should the value of k be in terms of A, T, and r?

Solution: First set up the variables.

```
restart;
A := 'A';  # the original amount borrowed.
a := 'a':  # the amount of money owed after t years
k := 'k':  # the  annual rate of repayment
t := 't':  # time measured in years
r := 'r':  # the annual interest rate on the loan
```

$$A := A$$

The differential equation which governs the rate of change of the money owed at time t is then

$$\frac{da}{dt} = ratein - rateout = ra - k$$

```
diffeq := diff(a(t),t) = r*a(t)-k;
```

$$diffeq := \frac{\partial}{\partial t}\, a(\,t\,) = r\, a(\,t\,) - k$$

Using dsolve to solve the initial value problem.

```
sol := dsolve({diffeq,a(0)=A},a(t));
```

$$sol := a(\,t\,) = \frac{k}{r} + e^{(\,r\,t\,)}\left(-\frac{k}{r} + A\right)$$

Getting a formula for this type of loan payback.

```
payback := unapply(simplify(solve(subs(t=T,rhs(sol)),k)),(T,A,r));
```

$$payback := (\,T, A, r\,) \to \frac{e^{(\,r\,T\,)}\, A\, r}{-1 + e^{(\,r\,T\,)}}$$

So for example if we borrowed 8000 dollars for three years at 10 percent compounded continuously, we would need to pay back

```
payback(3.,8000.,.1);
```

$$3086.636727$$

dollars per year. This would be done continuously, so that every instant we would be in the loan office making a very small payment on our loan.

Really it is not too realistic to pay back a loan continuously. Usually, regular payments are made on a monthly basis. So, the problem above should be recalculated.

```
A := 8000.:  T := 3.: r := .1: k := 'k':
for n from 1 to 12*T do
A := exp(r*1/12.)*A - k/12. od:
```

Here we have made 36 monthly payments on the loan, where k now is divided into 12 equal installments. The amount we owe on the loan is then, in terms of k,

```
A;
```

$$10798.87038 - 3.484030849\, k$$

Since this is supposed to be 0 after 36 payments, we can solve for k.

```
solve(A,k);
```

$$3099.533514$$

So there is only about a 13 dollar difference per year here. For some planning purposes the differential equation model for this situation will do nicely.

Problem 1. Make a payback function for the general case of borrowing A dollars for t years at r percent annual interest compounded continuously assuming that the loan is paid off in equal monthly installment.

Problem 2. Suppose you save 1000 dollars a year into an account which pays 10 percent interest compounded continuously. How much will the account contain after 10 years? Model this with a differential equation and use that to estimate the amount in the account. Then assume that the payments into the account are made in equal monthly installments and calculate the exact amount. Compare. Solve the more general problem.

6.4 Two Numerical Methods

6.4.1 Picard's Method

Picard's iterative method for solving a first order equation can be used to establish the existence and uniqueness theorem:

Theorem: If $f(x,y)$ and $f_y(x,y)$ are continuous on the rectangle $|x| < a$, $|y| < b$, then there is a unique solution $y = y(x)$ to $y'(x) = f(x, y(x))$, $y(0) = y0$ on some interval $|x| < h < a$.

The method consists of constructing a sequence of functions y_i which you then show converges uniformly to a solution on some interval about 0. The continuity of f_y is used to assure the uniqueness of the solution. Picard noticed that a solution to the initial value problem $y'(x) = f(x, y(x))$, $y(0) = y0$ also satisfies the integral equation

```
y(x) := y0 + Int(f(t,y(t)),t=0..x);.
```

$$y0 + \int_0^x f(t, y(t))dt = y0 + \int_0^x f(t, y(t))dt$$

As a first approximation to $y(x)$ let y_0 be the constant function $y0$. Then y_1 is defined by $y_1(x)$ is y0 + the integral of $f(t, y_0(t))$ over the interval 0 to x, and inductively, y_n is defined by

```
y_n(x) = y0 + Int(f(t,y_(n-1)(t)),t=0..x);
```

$$y\_n(x) = y0 + \int_0^x f(t, y\_(n-1)(t))dt$$

The hope is that as n gets large, the sequence y_n converges to the solution we seek.

For example, suppose $f(x,y) = (y+1)$, $y(0) = 1$. The solution to this linear ivp $y = 2e^x - 1$. Let's generate the first 5 terms of Picard's sequence and plot them along with the true solution to the problem.

```
f := (x,y) -> y+1;
```

$$f := (\, x, y\,) \to y + 1$$

```
diffeq := diff(y(x),x)=f(x,y(x));
sol := dsolve({diffeq,y(0)=0},y(x));
p := unapply(rhs(sol),x);   # the true solution
```

$$diffeq := \frac{\partial}{\partial x}\, y(\, x\,) = y(\, x\,) + 1$$

$$sol := y(\, x\,) = -1 + e^x$$

$$p := x \to -1 + e^x$$

```
p.0 := x -> 1;
for i from 1 to 5 do
p.i := unapply(p.(i-1)(0) + int(f(t,p.(i-1)(t)),t=0..x),x) od;
```

$$p0 := 1$$

$$p1 := x \to 1 + 2\,x$$

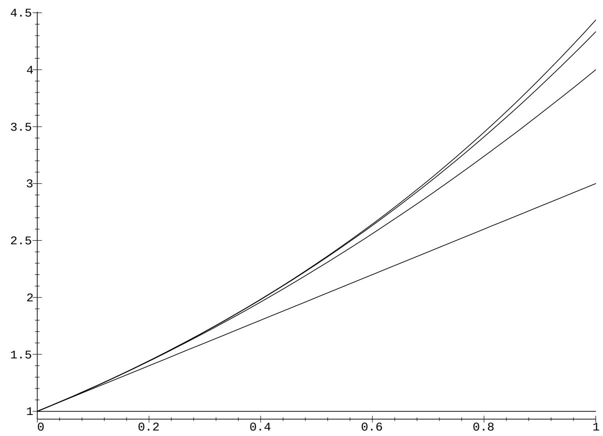
$$p2 := x \to 1 + 2\,x + x^2$$

$$p3 := x \to 1 + 2\,x + \frac{1}{3}\,x^3 + x^2$$

$$p4 := x \to 1 + 2\,x + \frac{1}{12}\,x^4 + x^2 + \frac{1}{3}\,x^3$$

$$p5 := x \to 1 + 2\,x + \frac{1}{60}\,x^5 + x^2 + \frac{1}{3}\,x^3 + \frac{1}{12}\,x^4$$

```
plot({seq(p.i,i=0..4)},0..1,color=black)],title='Picard approximates')');
```



Problem 1. What happens to the sequence of Picard approximates when the initial term y_0 is changed? For example, change y.0 above to the cosine function. Then recompute and plot the sequence of approximates. What would be one advantage of sticking with $y_0 = y0$ as the initial approximate?

Problem 2. How accurate an approximation to the true solution is the 4th Picard approximate to `y' = 2*x*(1+y)`, `y(0)=1`, starting with $y_0 = 1$?

Problem 3. For the ivp $y' = \sin(x + y)$, $y(0) = 0$, try to use Picard's method to generate an approximate solution. What happens? Does this point up a limitation of this method as a practical method of solving differential equations?

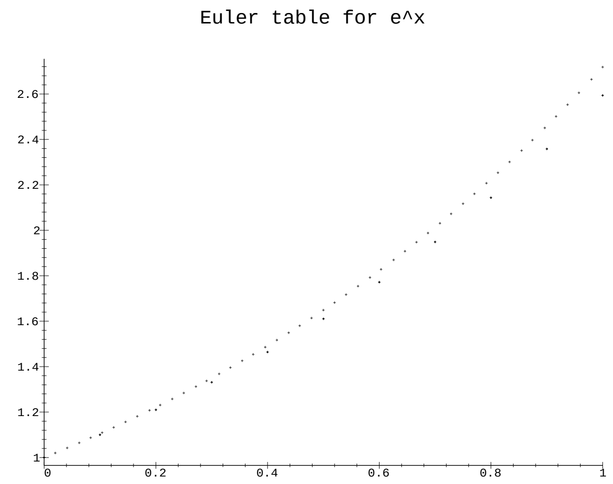
6.4.2 Euler's Method for solving an initial value problem.

Recall from Calculus I the idea of the differential. Given a point x_0 at which the value $f(x_0)$ and derivative $f'(x_0)$ of a function f are known, the differential $df = f'(x_0)dx$ of f at x_0 provides a good approximation to the change in f, $f(x_0+h) - f(x_0)$ when h is small. Put another way, $f(x_0) + f'(x_0)h$ is a good approximation to $f(x_0 + h)$ when h is small. Euler's method is an extension of this idea, and is used to generate a table of approximate values of the solution to the initial value problem $y' = g(x, y)$, $y(x_0) = y_0$. First choose an increment h and approximate the value of $y(x_0 + h)$ with $y_1 = y(x_0) + g(x_0, y_0) * h$. To approximate the value of $y(x_0 + 2h)$, we would add to y_1 the differential $g(x_0 + h, y1)h$. In general to estimate $y(x_0 + n * h)$ we would compute $y_n = y_{n-1} + g(x + i * h, yn - 1)h$. For example, if $g(x, y) = y$ and $y(0) = 1$, the solution is $y(x) = \exp(x)$. Using an increment of $h = .1$ and Euler's method to get approximate values for $\exp(i/10)$ as i go from 1 to 10, we obtain a table of approximate values for the exponential function.

```
g := (x,y) -> y:
x.0 := 0:
y.0 := 1:
h := .1:
tab := x.0,y.0:
for i from 1 to 10 do
 x.i := x.(i-1) + h:
 y.i := y.(i-1)+h*g(x.(i-1),y.(i-1)):
 tab := tab, x.i,y.i:
od:
tab;
```

$$0, 1, .1, 1.1, .2, 1.21, .3, 1.331, .4, 1.4641, .5, 1.61051, .6, 1.771561, .7, \\ 1.9487171, .8, 2.14358881, .9, 2.357947691, 1.0, 2.593742460$$

plotting these table point together with the exponential function itself we see a pretty good fit, with about a 5 percent error developing out near 1 (2.72 - 2.59)/2.72



Problem 1. What percentage error is incurred when e^1 is estimated using Euler's method starting at $x_0 = 0$, $y_0 = 1$ using an increment of .05? .01? Plot each table of values as was done above. What does it appear that the tables are converging to?

Problem 2. Use Euler's method to construct a table of values for the solution to `y' = sin(x+y)`, `y(0)=1`, over the interval 0 to π. Use increments of h = .1, h = .05, and h = .025 and plot the tables as above. Do you see some convergence of the tables? Compare this with the results you get from working with Picard's method.

Problem 3. Use Euler's method to investigate the solution to initial value problems where the function g is not continuous. for example suppose `y'(x) = g(x,y) = round(x+y)` and `y(0) = 0`. Plot several tables and comment on the nature of the solution. Change the initial value to $y(0) = 1$ and repeat the experiment. Extra credit: animate the plotting of the tables.

6.5 Chaos

6.5.1 The Logistic Equation

The logistic difference equation is the equation $x_{n+1} = f(x_n)$, where $f(x) = r*x*(1-x)$, r a fixed number, called the *growth rate factor.* Starting from an initial value x_0, the equation can be used to generate a sequence x_0, x_1, ... by iteration. This equation was being used to model the population changes in successive generations of grasshoppers by Robert May in the early 1970's when he discovered that the population changes from generation to generation exhibited 'chaotic behaviour' as the growth rate r increased past around 3.57. In this Maple session,we develop some procedures for doing a visual investigation of the logistic equation. The 'staircase diagram' associated with a sequence x_0, x_1, ... is the picture you get when the set of points $(x_0, 0)$, (x_0, x_1), (x_1, x_1), (x_1, x_2), (x_2, x_2), ..., (x_n, x_n), (x_n, x_{n+1}), ... is plotted using `style = line`.

```
restart;
```

First, we develop an input cell which returns the sequence of points we want to plot in the stairstep diagram.

```
f := x -> r*x*(1-x);
m := 20;       # the number of steps
u0 := .72;     # the starting point
r := 3.63;     # the growth rate
u1 := f(u0):   # begin the iteration
stairstep := u0,0,u0,u1,u1,u1;   # begin stairstep
for j from 1 to m do         # make
 u0 := u1; u1 := f(u0);      # the stairstep
stairstep := stairstep,u0,u1,u1,u1 od:
```

$$f := x \to r\,x\,(1-x)$$

$$m := 20$$

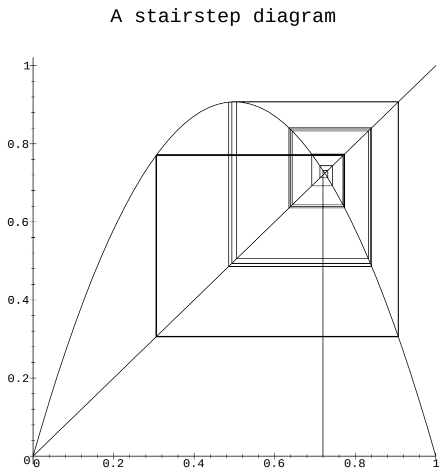
$$u0 := .72$$

$$r := 3.63$$

$$stairstep := .72, 0, .72, .731808, .731808, .731808$$

Now in order to make good use of this, we need to plot the graphs of $y = x$ and $y = rx(1-x)$ along with the stairstep.

```
plot({f,x->x,[stairstep]},0..1,style=LINE,
scaling=CONSTRAINED,
color=black,title=`A stairstep diagram`);  # plot the stairstep
```



Once we get this worked up, we can use the cell to define a word `stairs` which returns the plot of a set of stairs for a given rate r, length m, and starting value s. (It turns out that one can define functions easier within procedures by using `unapply` , rather than an arrow operator.)

```
stairs := proc(r,m,s)
local f,u0,x,u1,stairstep,j;
f := unapply(r*x*(1-x),x);
u0 := s;
u1 := f(u0):
stairstep := u0,0,u0,u1,u1,u1;   # begin stairstep
for j from 1 to m do             # make
u0 := u1; u1 := f(u0);           #   the stairstep
stairstep := stairstep,u0,u1,u1,u1 od:
[stairstep]
end;
```

testing our procedure out –

```
k := .5:  # growth rate.
m := 10: # number of stairs.
st := .75:  # starting point.
```

6.5.2 Problems

Problem 1. Plot the sequence of generations for k taking on values .5 .75, 1, 1.25, 1.5, 2.5, and 3. Leave m and st unchanged for the moment. Note any changes in the sequences which take place. Is it helpful to have the identity function $x->x$ plotted along with the growth curve $x->kx(1-x)$?

Now let's do some animation. To see what happens to the sequence of generations when we increase the growth rate, we can make a set of plots with the growth rate increasing and then display them in sequence.

```
f := x -> x*k*(1-x):
pl.i := plot({x->x,f,stairs(k,10+3*i,.3)},0..1,
   style=line,color=black) od:
plots[display]([seq(pl.i,i=1..8)],insequence=true);
```

Problem 2. Run this movie several times, observing the changes in the sequence of populations. In particular, describe what happens as the growth rate k gets up near 3.5? Change the starting point .3 to something else and see if the behaviour changes. Would you describe the behavior as 'chaotic'?

6.6 Second order linear equations

One of the simplest second order equations is: $y'' + y = 0$ or $y'' = -y$ Two basic solutions: $y1 = \cos(x)$ and $y2 = \sin(x)$. Any linear combination of y1 and y2 is also a solution:

$(yh)'' + yh = (sy1 + ty2)'' + (sy1 + ty2) == s(y1'' + y1) + t(y2'' + y2) = s0 + t0 = 0.$

Theorem: If y is any solution to $y'' + y = 0$, then y is a linear combination of sin and cos.

Proof: Given any initial conditions y(0) and y'(0), we can find a linear combination of sin and cos which satisfies the equation and has those values. The theorem follows from the uniqueness of the solution.

Let's look at how maple solves this equation: First set up the equation.

```
diffeq := diff(y(x),x,x) + y(x) =0;
```

$$diffeq := \left(\frac{\partial^2}{\partial x^2} y(\,x\,)\right) + y(\,x\,) = 0$$

Then set up the initial conditions.

```
inits := y(0)=y0, D(y)(0)=yp0;
```

$$inits := y(\,0\,) = y0, D(\,y\,)(\,0\,) = yp0$$

Solve the equation, using `dsolve`.

```
sol := dsolve({diffeq,inits},y(x));
```

$$sol := y(\,x\,) = y0\,cos(\,x\,) + yp0\,sin(\,x\,)$$

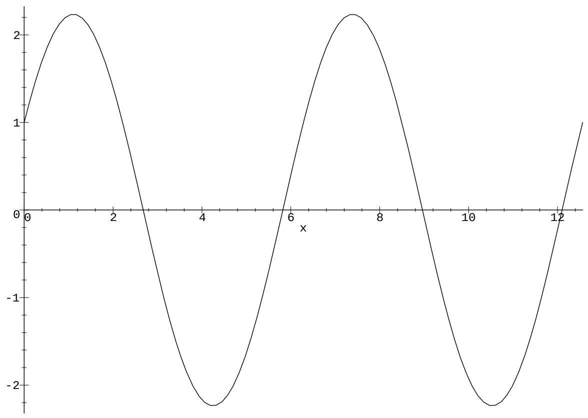
Turn sol into a function of x, together with the parameters y0 and yp0

```
f := unapply(rhs(sol),(y0,yp0,x));
```

$$f := (\,y0, yp0, x\,) \rightarrow y0\,cos(\,x\,) + yp0\,sin(\,x\,)$$

To plot the solution to $y'' + y = 0$, $y(0) = 1, y'(0) = 2$, we would

```
plot(f(1,2,x),x=0..4*Pi);
```



Problem. Animate the solution to the equation, keeping y(0) at 1, but letting y'(0) go from -1 to 1. Describe what you see.

Solution:

```
plots[animate](f(1,yp0,x),x=0..4*Pi,yp0=-1..1);
```

Upon examination of the movie, we see the solution is a simple harmonic motion, which always passes through (0,1), but where the slope of the solution at (0,1) gradually changes from - 1 to 1.

6.6.1 Problems

Problem 1. Find the solution to $y'' = -5y$, $y(0) = 3$, $y'(0) = 4$. Find the maximum and minimum values of this solution. What if the 5 were replaced by a positive number mu? Express the solution of x and the parameter μ. Then animate the solution as mu changes from 1 to 4.

An equally simple second order equation is $y'' - y = 0$, or $y'' = y$ Here there are two solutions also, namely $y1 = \exp(x)$ and $y2 = \exp(-x)$. As before, any linear combination of these solutions is also a solution, and any solution to the equation is a linear combination of these functions.

Problem 2. Use Maple to solve the equation $y'' = ky$, $y(0) = 1$, $y'(0) = 2$ where k is a positive number.

Problem 3. Find the function $y = f(x)$, such that $f''(x) = f(x)$, $f(0) = 10$, and $f'(0) = 2$. When does $f(x) = 5$?

The more general second order equation with constant coefficients looks like

```
a y''  + b y' + c y = 0
```

This can be solved by assuming there is an exponential solution. $y = \exp(rx)$. Then $y' = r\exp(rx)$, and $y'' = r^2\exp(rx)$. plugging these into the equation, get $\exp(rx)(ar^2 +$

$br + c) = 0$. Since $\exp(rx)$ is never 0, the roots of the characteristic equation yield the exponential solutions.

Problem 4. Find the solution to the initial value problem $12y'' + 5y' - 2y = 0$, $y(0) = 3$, $y'(0) = 1$. Then find $y(2)$. Also find when $y(x) = 10$.

6.7 Reduction of order:

Given a 2nd order homogeneous linear differential equation $p(x)y'' + q(x)y' + r(x)y = 0$. If you already have found one solution y_1 somehow (for example the equation has constant coefficients and the characteristic equation has only one root), then you can search for another solution y_2 by letting $y_2 = vy_1$ and substituting that in for y in the given equation. After simplification you always get a second order linear equation in v with no v term. So, you can solve for v in two steps: First let $w = v'$. Then $w' = v''$ can be expressed in terms of w and x. After solving this first order equation for w, then solve the first order equation $v' = w$ for v. At that point you have found y_2. Let's test this algorithm out on a contrived example.

```
restart;
```

Restarting and defining the differential equation.

```
diffeq := x*diff(y(x),x,x) -diff(y(x),x) +4*x^3*y(x) =0;
```

$$diffeq := x\left(\frac{\partial^2}{\partial x^2}y(x)\right) - \left(\frac{\partial}{\partial x}y(x)\right) + 4\,x^3\,y(x) = 0$$

Two linearly independent solutions to this equation are $\sin(x^2)$ and $\cos(x^2)$. Let's take y_1 to be $\sin(x^2)$ and see what y_2 reduction of order yields.

```
y1 := x -> sin(x^2);
```

$$y1 := x \to sin(x^2)$$

Now we are looking for an independent solution y_2 of the form y_1 times an unknown function of x.

```
y2 := x -> v(x)*y1(x);
```

$$y2 := x \to v(x)\,y1(x)$$

Now substitute this into the differential equation.

```
subs(y(x)=y2(x),diffeq);
```

$$x\left(\frac{\partial^2}{\partial x^2}v(x)\,sin(x^2)\right) - \left(\frac{\partial}{\partial x}v(x)\,sin(x^2)\right) + 4\,x^3\,v(x)\,sin(x^2) = 0$$

and simplify. Note that there is a v" term and two v' terms, but no v term.

```
simplify(");
```

$$x\left(\frac{\partial^2}{\partial x^2}v(x)\right)sin(x^2) + 4\left(\frac{\partial}{\partial x}v(x)\right)cos(x^2)\,x^2 - \left(\frac{\partial}{\partial x}v(x)\right)sin(x^2) = 0$$

For visual purposes, let's collect the v' terms,

```
collect(",diff(v(x),x));
```

$$(4\,cos(\,x^2\,)\,x^2 - sin(\,x^2\,))\,\left(\frac{\partial}{\partial x}\,v(\,x\,)\right) + x\,\left(\frac{\partial^2}{\partial x^2}\,v(\,x\,)\right)\,sin(\,x^2\,) = 0$$

and start the order reduction by substituting w(x) in for v'(x).

```
subs(diff(v(x),x)=w(x),");
```

$$(4\,cos(\,x^2\,)\,x^2 - sin(\,x^2\,))\,w(\,x\,) + x\,\left(\frac{\partial}{\partial x}\,w(\,x\,)\right)\,sin(\,x^2\,) = 0$$

This is a linear differential equation in w(x), and we could continue to solve this 'by hand', but lets use dsolve since we already know how that goes anyway.

```
dsolve(",w(x));
```

$$w(\,x\,) = -\frac{x\,\_C1}{-1 + cos(\,x^2\,)^2}$$

There is a constant of integration that arises. Since we only want one v, we can just set $\_C1 = 1$.

```
subs(_C1=1,");
```

$$w(\,x\,) = -\frac{x}{-1 + cos(\,x^2\,)^2}$$

Now turn w into a function.

```
w := unapply(rhs("),x);
```

$$w := x \to -\frac{x}{-1 + cos(\,x^2\,)^2}$$

Solve the equation v' = w by integration.

```
v := unapply(int(w(x),x),x);
```

$$v := x \to \frac{1}{4}\,tan\left(\frac{1}{2}\,x^2\right) - \frac{1}{4}\,\frac{1}{tan\left(\frac{1}{2}\,x^2\right)}$$

Checking to see if our earlier definition of y_2 is updated –

```
y2(x);
```

$$\left(\frac{1}{4}\,tan\left(\frac{1}{2}\,x^2\right) - \frac{1}{4}\,\frac{1}{tan\left(\frac{1}{2}\,x^2\right)}\right)\,sin(\,x^2\,)$$

It is, and so the general solution to the equation can be set up as a linear combination of the solution y_1 we started with and the solution y_2 we constructed by the method called 'reduction of order'.

```
y := unapply(c1*y1(x)+c2*y2(x),(c1,c2,x));
```

$$y := (\,c1, c2, x\,) \to c1\,sin(\,x^2\,) + c2\,\left(\frac{1}{4}\,tan\left(\frac{1}{2}\,x^2\right) - \frac{1}{4}\,\frac{1}{tan\left(\frac{1}{2}\,x^2\right)}\right)\,sin(\,x^2\,)$$

I notice y_2 is a fairly ugly looking function. It probably will simplify some.

```
sim := simplify(y2(x),trig);
```

$$sim := -\frac{1}{4}\frac{sin(\, x^2\,)\left(-1+2\,cos\left(\frac{1}{2}\,x^2\right)^2\right)}{cos\left(\frac{1}{2}\,x^2\right)\,sin\left(\frac{1}{2}\,x^2\right)}$$

Certainly not as simple as $\cos(x^2)$! But looks are sometimes deceiving –

```
sim := combine(numer(sim),trig)/combine(denom(sim),trig);
```

$$sim := -\frac{1}{4}\frac{sin(\, 2\,x^2\,)}{sin(\, x^2\,)}$$

```
sim := expand(sim,trig);
```

$$sim := -\frac{1}{2}\,cos(\, x^2\,)$$

So reduction of order returns $y_2 = -1/2\cos(x^2)$.

6.7.1 Problems

Problem 1. Use y_1 and y_2 to solve the initial value problem

```
diffeq := x*diff(y(x),x,x) -diff(y(x),x) +4*x^3*y(x)=0;
inits :=  y(2) = 1, D(y)(2) = 1;
```

Then plot it over the interval 1 .. 6.

Problem 2. Let y_1 be $\sin(x^2)+\cos(x^2)$. This will be a solution to the diffeq above. (Why?) Use reduction of order to find an independent solution y_2.

6.8 Euler equations.

Let a and b be real numbers. The equation

$$x^2y''(x)+axy'(x)+by(x)=0$$

is called an Euler equation. These equations can be transformed to equations with constant coefficients by making a change of variable. $x=\exp(z)$, and hence $z=\ln x$. Note that with these substitutions $\dfrac{dx}{dz}=\exp(z)=x$ and $\dfrac{dz}{dx}=1x$.

Now y is a function of z and

$$y'=\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}\frac{1}{x},$$

by the chain rule.

Also

$$y''=\frac{d(\frac{dy}{dx})}{dx}=\frac{d((\frac{dy}{dz})\frac{1}{x}}{dx}=\frac{d(\frac{dy}{dz}\frac{1}{x})}{dz}\frac{dz}{dx}=(\frac{d^2y}{dz^2}\frac{1}{x}+\frac{dy}{dz}(\frac{-1}{x^2}\frac{dx}{dz}))(\frac{1}{x}),$$

using the product rule twice. Continuing the series of equations,

$$= \frac{d^2y}{dz^2}\frac{1}{x^2} - \frac{dy}{dz}\frac{1}{x^2}.$$

Plugging these into the Euler equation, we get

$$\frac{dy^2}{dz^2} - \frac{dy}{dz} + a\frac{dy}{dz} + by = 0.$$

Let's implement this method using Maple, without using `dsolve`.

First restart and define the differential equation.

```
restart;
diffeq :=  x^2 *diff(y(x),x,x) + a*x* diff(y(x),x) + b* y(x) = 0;
```

$$diffeq := x^2\left(\frac{\partial^2}{\partial x^2}\,y(\,x\,)\right) + a\,x\left(\frac{\partial}{\partial x}\,y(\,x\,)\right) + b\,y(\,x\,) = 0$$

Make the substitution and get a new differential equation.

```
ypp := diff(y(x),x,x)=diff(y(z),z,z)*1/x^2-diff(y(z),z)*1/x^2:
yp := diff(y(x),x) = diff(y(z),z)*1/x:
diffeq1 := subs({ypp, yp,y(x)=y(z)},diffeq);
```

$$diffeq1 := x^2\left(\frac{\frac{\partial^2}{\partial z^2}\,y(\,z\,)}{x^2} - \frac{\frac{\partial}{\partial z}\,y(\,z\,)}{x^2}\right) + a\left(\frac{\partial}{\partial z}\,y(\,z\,)\right) + b\,y(\,z\,) = 0$$

Now simplify this.

```
diffeq1 :=simplify(diffeq1);
```

$$diffeq1 := \left(\frac{\partial^2}{\partial z^2}\,y(\,z\,)\right) - \left(\frac{\partial}{\partial z}\,y(\,z\,)\right) + a\left(\frac{\partial}{\partial z}\,y(\,z\,)\right) + b\,y(\,z\,) = 0$$

Collect the $dydz$ terms for visual purposes.

```
diffeq1 := collect(diffeq1,diff(y(z),z));
```

$$diffeq1 := (\,-1 + a\,)\left(\frac{\partial}{\partial z}\,y(\,z\,)\right) + \left(\frac{\partial^2}{\partial z^2}\,y(\,z\,)\right) + b\,y(\,z\,) = 0$$

This is a linear 2nd order equation with constant coefficients. We could finish this without `dsolve`, couldn't we?

```
sol := dsolve({diffeq1},y(z));
```

$$sol := y(\,z\,) = \_C1\,e^{\left(1/2\left(1-a+\sqrt{1-2\,a+a^2-4\,b}\right)z\right)} + \_C2\,e^{\left(-\,1/2\left(-1+a+\sqrt{1-2\,a+a^2-4\,b}\right)z\right)}$$

Transform the solution back to a function of x.

```
y := unapply(subs(z=ln(x),rhs(sol)),x);
```

$$y := x \to \_C1\,e^{\left(1/2\left(1-a+\sqrt{1-2\,a+a^2-4\,b}\right)ln(\,x\,)\right)} + \_C2\,e^{\left(-\,1/2\left(-1+a+\sqrt{1-2\,a+a^2-4\,b}\right)ln(\,x\,)\right)}$$

Let's set up a function of the four parameters a,b,$\_C1$, $\_C2$, and the variable x.

```
yg := unapply(simplify(y(x)),(a,b,_C1,_C2,x));
```

$$yg := (a, b, \_C1, \_C2, x) \to \_C1\, x^{\left(1/2-1/2\,a+1/2\,\sqrt{1-2\,a+a^2-4\,b}\right)} + \_C2\, x^{\left(1/2-1/2\,a-1/2\,\sqrt{1-2\,a+a^2-4\,b}\right)}$$

6.8.1 Problems

Problem Solve the Euler equation $x^2y'' + axy' + 5y = 0$.

i) For what values of a are the fundamental solutions both real-valued functions?

ii) Let $a = 1$. Find a real solution to the Euler equation above. Hint: try a linear combination of the complex fundamental solutions yielded by the Euler change of variable.

6.9 Undetermined coefficients:

This method of finding a particular solution to a nonhomogeneous linear equation with constant coefficients usually works when the right hand side is a linear combination of polynomials, exponential functions, and sines and cosines and their products. We then write down a trial solution consisting of a linear combination of functions of the same type with coefficients undetermined. Then simply substitute the trial solution into the equation and collect like terms. At that point, we get some equations in the unknown coefficients which if we can solve will then provide a solution. For example, to solve the ivp $y''+4y = x^2+3*e^x$, $y(0) = 0$, $y'(0) = 2$.

Set up the differential equation and set up the form of the particular solution yp.

```
restart;
eq1 := (D@@2)(y)(x)+4*y(x) - x^2 - 3*exp(x);
yp := x -> A + B*x + C*x^2 + D*exp(x);
```

$$eq1 := D^{(2)}(y)(x) + 4\,y(x) - x^2 - 3\,e^x$$

$$yp := x \to A + B\,x + C\,x^2 + D\,e^x$$

Substitute yp into the equation and evaluate.

```
subs(y=yp,eq1);
eval(");
```

$$D^{(2)}(yp)(x) + 4\,yp(x) - x^2 - 3\,e^x$$

$$2\,C + 5\,D\,e^x + 4\,A + 4\,B\,x + 4\,C\,x^2 - x^2 - 3\,e^x$$

Collect the coefficients of like terms.

```
collect(",[x,exp(x)]);
```

$$(4\,C-1)\,x^2 + 4\,B\,x + (5\,D-3)\,e^x + 2\,C + 4\,A$$

Solve for the undetermined coefficients by inspection: C = 1/4 , etc., put into yp.

```
yp := subs({C=1/4,B=0,D=3/5,A=1/4*(-1/2)},op(yp));
```

$$yp := x \to -\frac{1}{8} + \frac{1}{4}x^2 + \frac{3}{5}e^x$$

Solve the associated homogeneous equation by inspection. (characteristic equation)

```
yh :=  x ->C1* sin(2*x) + C2*cos(2*x);
```

$$yh := x \to C1\,sin(\,2\,x\,) + C2\,cos(\,2\,x\,)$$

Check our work by substituting yh+yp into the original equation and evaluating.

```
subs(y=yp+yh,eq1);
eval(");
```

$$D^{(\,2\,)}(\,yp + yh\,)(\,x\,) + 4\,(\,yp + yh\,)(\,x\,) - x^2 - 3\,e^x$$

$$0$$

Voila! We did good. Now use the given initial values to calculate C1 and C2.

```
sol := solve({(yh+yp)(0)=0,(D(yh+yp))(0)=2},{C1,C2});
```

$$sol := \left\{C2 = \frac{-19}{40}, C1 = \frac{7}{10}\right\}$$

Substitute these values into yg = yh + yp to get the solution to the IVP.

```
y := unapply(subs(sol,(yh+yp)(x)),x);
```

$$y := x \to -\frac{1}{8} + \frac{1}{4}x^2 + \frac{3}{5}e^x + \frac{7}{10}sin(\,2\,x\,) - \frac{19}{40}cos(\,2\,x\,)$$

We can see that the oscillatory part of the solution will be overwhelmed by the exponential part as x gets large, and by the quadratic term as x goes to -infinity.

```
plot(y,-2..2,-2..2);
```

6.9.1 Problems

Problem. Use undetermined coefficients to solve these. Plot.

1. $y'' - 5y' + 6y = e^x\cos(2x) + e^{2x}\sin(2x) + x + 2x^2$, $y(0) = 1$,$y'(0) = -1$.

2. $y'' + 3y' + 2y = e^x(x^2+1)\sin(2x) + 3e^{-x}\cos(x) + 4e^x$, $y(0) = 2/3$, $y'(0) = -1/2$.

6.10 Variation of parameters.

This method of finding a particular solution to a nonhomogeneous linear equation will always work in theory, provided you have a basis for the solution space of the corresponding homogeneous equation. Let's illustrate the method with an example.

Problem: Find the function $y(x)$ such that $y'' + y = tan$ and $y(0) = 1$, $y'(0) = -1$.

```
restart;
eq := (D@@2)(y)(x) + y(x) - tan(x);
```

$$eq := D^{(2)}(y)(x) + y(x) - tan(x)$$

Get two linearly independent solutions (here by the characteristic equation the solutions would be $\sin(x)$ and $\cos(x)$). and form the particular solution yp as a linear combination with nonconstant coefficients.

```
yp := u1*cos + u2*sin;
```

$$yp := u1\, cos + u2\, sin$$

There will be lots of solutions to this, simply because there are lots of particular solutions to the equation. So we are free to impose a condition on yp and it was the insight of LaGrange that led him to assume that $D(u1)\cos + D(u2)\sin = 0$.

```
cond1 :=   'D'(u1) *cos = - 'D'(u2)*sin ;
```

$$cond1 := D(\,u1\,)\, cos = -D(\,u2\,)\, sin$$

So that when we take the derivative of yp, we get only two terms instead of four.

```
ypp := subs({cond1},D(yp));
```

$$ypp := -u1\, sin + u2\, cos$$

A second condition on yp comes from the assumption that yp satisfies the equation.

```
cond2 := D(ypp) + yp - tan;
```

$$cond2 := -D(\,u1\,)\, sin + D(\,u2\,)\, cos - tan$$

This gives us two linear equations in the derivatives of the two unknown functions u1 and u2. We can solve for $u1'$ and $u2'$ since cos and sin are linearly independent and so have a nonzero wronskian of $\sin^2 + \cos^2 = 1$.

```
sol := solve({cond1,cond2},{D(u1),D(u2)});
```

$$sol := \left\{ D(\,u1\,) = -\frac{tan\, sin}{cos^2 + sin^2},\, D(\,u2\,) = \frac{cos\, tan}{cos^2 + sin^2} \right\}$$

Make a substitution.

```
sol2:=subs(cos^2+sin^2=1,sol);
```

$$sol2 := \{\, D(\,u1\,) = -tan\, sin,\, D(\,u2\,) = cos\, tan\,\}$$

Now we can integrate to get u1 and u2.

```
u1p := rhs(sol2[1]);
```

$$u1 := sin(\,x\,) - ln(\,sec(\,x\,) + tan(\,x\,)\,)$$

```
u2p := rhs(sol2[2]);
```

$$u2p := cos\, tan$$

```
u1 := unapply(int(u1p(x),x),x);
```

$$u1 := x \to sin(\,x\,) - ln(\,sec(\,x\,) + tan(\,x\,)\,)$$

```
u2 := unapply(int(u2p(x),x),x);
```

$$u2 := -cos$$

Now the homogeneous solution is

```
c1 := x->c1:  c2 := x->c2:
yh :=c1*sin+c2*cos;
```

$$yh := c1\, sin + c2\, cos$$

So the general solution is

```
yg := (yp + yh);
yg(x);
```

$$yg := u1\, cos - cos\, sin + c1\, sin + c2\, cos$$

$$(\, sin(\, x\,) - ln(\, sec(\, x\,) + tan(\, x\,)\,)\,)\, cos(\, x\,) - cos(\, x\,)\, sin(\, x\,) + c1\, sin(\, x\,) + c2\, cos(\, x\,)$$

Checking this by plugging it into the original equation and simplifying – (it woiks!)

```
simplify(eval(subs(y=yg,eq)),trig);
```

$$0$$

Now find the value of c1 and c2 which satisfy the initial values.

```
sol := solve({yg(0)=2,D(yg)(0)=-3},{c1,c2});
```

$$sol := \{\, c2 = 2, c1 = -2\,\}$$

```
subs(sol,yg(x));
```

$$(\, sin(\, x\,) - ln(\, sec(\, x\,) + tan(\, x\,)\,)\,)\, cos(\, x\,) - cos(\, x\,)\, sin(\, x\,) - 2\, sin(\, x\,) + 2\, cos(\, x\,)$$

```
plot(subs(sol,yg(x)),x=-1..1.4);
```

6.10.1 Problems

Solve these problems using variation of parameters.

1. $y'' + 2y' - y = \sin(2x)$, $y(0) = 1$, $y'(0) = -2$.

2. $x^2y'' - 2y = 3x^2 - 1$, $y1 = x^2$, $y2 = 1/x$.

6.11 Spring problems

6.11.1 Hooke's Law

Suppose you have a weight mg which you hang at the end of a spring. It will elongate the spring a certain amount L. Within limits, this amount is proportional to the force of gravity acting on the mass. So kL = mg. The value k is called the spring constant and can be obtained with one measurement. Now if we displace the weight a certain amount u(0) from the rest position (by convention, down is positive) and impart a certain initial vertical velocity u'(0) to the weight, then the motion of the weight on the end of the spring ought to be completely determined by Newton's laws of gravitation. Setting up the parameters:

```
m = mass of the weight
L = elongation at rest
u = u(t) = displacement = 0 at rest at t = 0.  down is positive.
k = spring constant.
mu =  resistance constant
F(t) = external driving  force.
```

By Newton, the force on the weight at time t, $F = mu''(t)$, will the sum of the force of gravity, $F_g = mg$, the force exerted by the spring $F_s = -k(L + u(t))$, the force of resistance which we assume is proportional to the velocity of the weight, $F_r = -\mu u'(t)$, and an external driving force $F(t)$. This provides us with a second order differential equation describing the motion of the system over time.

$$F = F_g + F_s + F_r + F(t)$$

which is also

$$mu''(t) = mg - k(L + u(t)) - \mu u'(t) + F(t).$$

Since kL = mg, the equation simplifies to $mu''(t) + \mu u'(t) + ku(t) = F(t)$. We want to make sure that the various kinds of motion we can imagine in advance can actually occur and relate that to the values of the parameters of the equation: m, μ, and k. Also we want to investigate how changing the initial values u(0) and u'(0) can affect the solution. In this worksheet, we will only look at the homogeneous case, when there is no external driver to the system, that is, $F(t) = 0$.

First set up the homogeneous equation.

```
restart;
spring  := m*diff(u(t),t,t) + mu*diff(u(t),t) + k*u(t) = 0;
```

$$spring := m\left(\frac{\partial^2}{\partial t^2}u(\,t\,)\right) + \mu\left(\frac{\partial}{\partial t}u(\,t\,)\right) + k\,u(\,t\,) = 0$$

The characteristic equation for this is:

```
char := m*x^2 + mu*x + k;
```

$$char := m\,x^2 + \mu\,x + k$$

and the solutions to the characteristic equation are:

```
csol := solve(char,x);
```

$$csol := \frac{1}{2}\,\frac{-\mu + \sqrt{\mu^2 - 4\,m\,k}}{m},\ \frac{1}{2}\,\frac{-\mu - \sqrt{\mu^2 - 4\,m\,k}}{m}$$

From this we can write down the general solution to the homogeneous equation depending on the sign of the discriminant $\mu^2 - 4km$.

Case 1 if the discriminant is positive, then two independent solutions are `y1 = exp(csol[1]*t)` and `y2 = exp(csol[2]*t)`.

```
y1 :=  t ->exp(csol[1]*t) ;   y2 := t->exp(csol[2]*t);
```

$$y1 := t \to e^{(csol_1\, t)}$$

$$y2 := t \to e^{(csol_2\, t)}$$

Case 2. If the discriminant is 0 then y1 is still good but `y2 = t*y1`.

```
y2 := t->t*exp(csol[1]*t);
```

$$y2 := t \to t\, e^{(csol_1\, t)}$$

Case 3. If the discriminant is negative then

```
y1  :=  t->exp(-mu/(2*m)*t)*cos(sqrt(k/m-(mu/(2*m))^2)*t);
y2  :=  t->exp(-mu/(2*m)*t)*sin(sqrt(k/m-(mu/(2*m))^2)*t);
```

$$y1 := t \to e^{\left(-1/2\, \frac{\mu\, t}{m}\right)}\, cos\left(sqrt\left(\frac{k}{m} - \frac{1}{4}\,\frac{\mu^2}{m^2}\right)\, t\right)$$

$$y2 := t \to e^{\left(-1/2\, \frac{\mu\, t}{m}\right)}\, sin\left(sqrt\left(\frac{k}{m} - \frac{1}{4}\,\frac{\mu^2}{m^2}\right)\, t\right)$$

In any case, the general solution to the homogeneous equation is a linear combination y1 and y2

```
yg := c1*y1 + c2*y2;
```

$$yg := c1\, y1 + c2\, y2$$

For our purposes, we can assume that the units for mass are scaled so that the mass of our weight is 1.

```
m := 1;
```

$$m := 1$$

6.11.2 Problems

Problem 1. $\mu = 0$ (Undamped motion) In this case the discriminant of the characteristic equation is $-4mk$ and its sign is determined by the sign of k.

```
mu := 0;
```

If $k > 0$, then the equation looks like $u'' + ku = 0$ and the solution looks like $yg = c1cos(sqrt(k)t) + c2sin(sqrt(k)t)$. We can set up the general solution as a function of the coefficients c1, c2, the spring constant k, and time t.

```
yg := (c1,c2,k,t) -> c1*cos(sqrt(k)*t) + c2*sin(sqrt(k)*t);
```

$$\mu := 0$$

$$yg := (\, c1, c2, k, t\,) \to c1\, cos(\, sqrt(\, k\,)\, t\,) + c2\, sin(\, sqrt(\, k\,)\, t\,)$$

Suppose $k = 1 = c2$. What happens to yg if c1 increases from 0 to 1? Look at the formula for yg above and try to visualize what happens. Then run the following movie and describe what happens.

```
plots[animate](yg(c1,1,1,t),t=0..10,c1=0..1,color=black);
```

Suppose we hold c1 = 1 = c2 and let k vary from 1/10 to 2. What happens to yg? Again, look at the formula for yg and try to predict what happens. Then make a movie and describe what happens. If k ¡ 0 then the discriminant is positive and the general solution yg looks like

```
yg := (c1,c2,k,t) -> c1*exp(sqrt(-k)*t) + c2*exp(-sqrt(-k)*t);
```

$$yg := (\, c1, c2, k, t\,) \to c1\, e^{(\,sqrt(\,-k\,)\,t\,)} + c2\, e^{(\,-sqrt(\,-k\,)\,t\,)}$$

It is a little hard to think of a spring with a negative spring constant, but if it exists it would move like this, if we hold c2=1=k and check yg over t=0..2 as c1 changes from 0 to 1.

```
plots[animate](yg(c1,1,-1,t),t=0..2,c1=0..1,color=black);
```

Problem 2. $\mu > 0$ (damped motion) If the resistance constant is positive then two situations can occur. If $\mu^2 < 4k$ then the discriminant is negative and the general solution looks like

```
mu := 'mu':
yg := (c1,c2,k,mu,t) -> c1*exp(-mu/(2*m)*t)*cos(sqrt(k/m-(mu/(2*m))^2)*t)+
          c2*exp(-mu/(2*m)*t)*sin(sqrt(k/m-(mu/(2*m))^2)*t);
```

$$yg := (\, c1, c2, k, \mu, t\,) \to$$
$$c1\, e^{\left(-1/2\,\frac{\mu\,t}{m}\right)}\, cos\left(sqrt\left(\frac{k}{m} - \frac{1}{4}\,\frac{\mu^2}{m^2}\right)\,t\right) + c2\, e^{\left(-1/2\,\frac{\mu\,t}{m}\right)}\, sin\left(sqrt\left(\frac{k}{m} - \frac{1}{4}\,\frac{\mu^2}{m^2}\right)\,t\right)$$

Investigate what happens to yg over t=0..6, if c1=c2=1, k = 4, and mu ranges from .1 to 3.9 Make a movie.

```
mu := 'mu';
```

$$\mu := \mu$$

If $\mu^2 > 4k$, then the discriminant is positive and the general solution looks like

```
yg := unapply(c1*exp(csol[1]*t)+c2*exp(csol[2]*t),(c1,c2,k,mu,t));
```

$$yg := (\, c1, c2, k, \mu, t\,) \to c1\, e^{\left(\left(-1/2\,\mu + 1/2\,\sqrt{\mu^2 - 4\,k}\right)\,t\right)} + c2\, e^{\left(\left(-1/2\,\mu - 1/2\,\sqrt{\mu^2 - 4\,k}\right)\,t\right)}$$

Investigate what happens to yg over t=0..6, if c1=c2=1, k = 4, and μ ranges from 4.1 to 6. Make a movie.

Problem 3. A mass that weighs 8 lbs stretches a spring 6 inches. The system is acted on by an external force $F(t) = 8\sin(8t)$ lb. If the mass is pulled down 3 inches and released, determine the position of the mass at any time. Find the first 4 times the velocity of the mass is 0.

Solution. Here $m = 8/32 = 1/4$, $F = 8 * \sin(8 * t)$, $k = 8/(1/2) = 16$. μ is 0.

```
restart; diffeq := (1/4) * diff( u(t),t,t) + 16 * u(t) = 8*sin(8*t);
```

$$diffeq := \frac{1}{4}\left(\frac{\partial^2}{\partial t^2}u(t)\right) + 16\,u(t) = 8\,sin(8\,t)$$

```
sol := rhs(dsolve({diffeq,u(0)=1/4,(D(u))(0)=0}, u(t)));
```

Lets simplify and write a polynomial in t here.

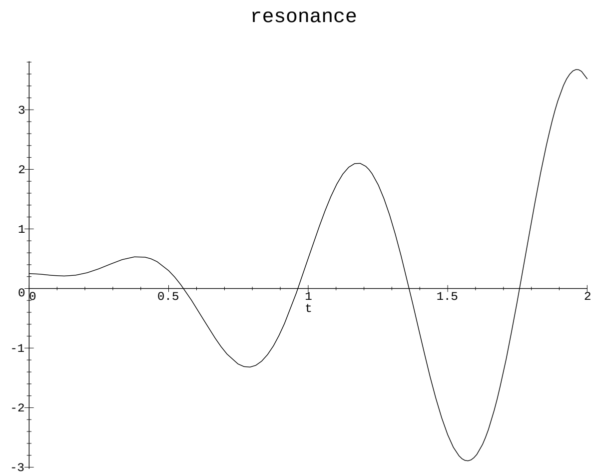
$$sol := -2\,cos(8\,t)\,t + \frac{1}{4}\,sin(8\,t) + \frac{1}{4}\,cos(8\,t)$$

```
displacement := collect(simplify( sol ),t);
```

$$displacement := -2\,cos(8\,t)\,t + \frac{1}{4}\,sin(8\,t) + \frac{1}{4}\,cos(8\,t)$$

So the solution can be viewed as a linear function of t with coefficients which are periodic with period no more than 2Pi.

```
plot( displacement, t=0..2,color=black,title=`resonance` );
```



Note that we have unbounded resonance. Now, to find the first four times where the velocity is zero, we can calculate and plot the velocity.

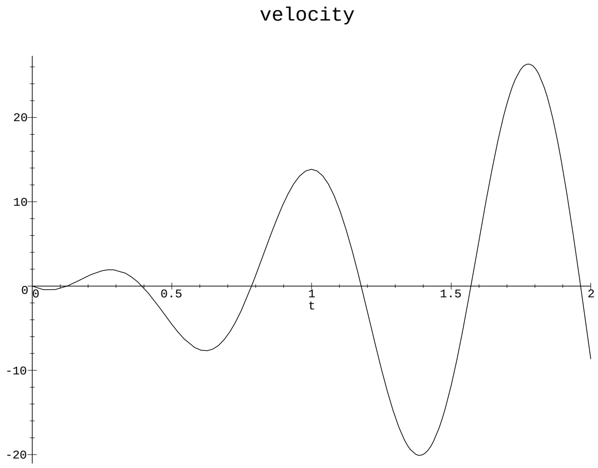
```
vel := collect(simplify(diff(displacement,t)),t);
```

$$vel := 16\,sin(8\,t)\,t - 2\,sin(8\,t)$$

```
printlevel:= 1;
```

$$printlevel := 1$$

```
plot( vel, t=0..2,color=black,title=`velocity` );
```



Upon examination of the graph, we see that the velocity is zero at t =0 (that was given) and then at three more times before t = 1. Now use fsolve to get the exact values.

```
first4zeros := 0,fsolve(velocity, t=0..(.3)),
 fsolve(velocity,t=(.3)..(.5)), fsolve(velocity,t=(.5)..1);
```

$$first4zeros := 0, .1250000000, .3926990817, .7853981634$$

Problem 4. A spring is stretched 6 inches by a mass that weighs 8 lbs. The mass is attached to a dashpot mechanism that exerts a constant damping coefficient of .25 lb-sec/ft. and is acted on by an external force of $F(t) = 4\cos(2t)$ lb.

Determine the motion of the mass, if the initial conditions are u(0) = 0 = u'(0).

6.12 Another spring problem.

Problem: Suppose you have an undamped spring with spring constant = 5 being driven by the 50 per cent square wave with period 2 and amplitude 2. Model the motion over the time interval 0 to 10.

A solution: Use the IVP $y'' + 5y' = 2SQW(t, 50, 2) = K(t)$, $y(0) = 0$, $y'(0) = 0$. If we use Variation of Parameters to find a particular solution what do we get?

First get fundamental solutions y1 and y2 to the homogeneous equation: $y1 = \sin(\sqrt{5}t)$ and $y2 = \cos(\sqrt{5}t)$. Then form the general solution $y = c1y1 + c2y2 + u1y1 + u2y2$ where $u1 = -\int y2K/W(y1, y2)$ and $u2 = \int y1K/W(y1, y2)$. The only problem with this is the integrations can't be carried out symbolically over the entire interval 0..10 at once, because K is this squarewave function. This means that the solution that is returned by Maple is expensive to evaluate and you get crummy plots from it. Here, I'll show you.

First let's program the basic square wave.

```
sqwv := proc(t)
 local s;
```

```
 if not type(evalf(t),float) then RETURN('sqwv(t)') fi;
 s := t-floor(t);
 if s < 1/2 then 1 else 0 fi
 end;
sqwv := proc(t)
        local s;
            if not type(evalf(t),float) then RETURN('sqwv(t)') fi;
            s := t-floor(t);
            if s < 1/2 then 1 else 0 fi
        end

plot(sqwv,-10..10);
```

Now we can scale the amplitude by multiplication.

```
G := 2*sqwv;
```

$$G := 2\,sqwv$$

```
plot({G,sqwv},-3..3);
```

And we can change the period by a linear change of variable.

```
K := 2*sqwv@(t-> t/2);
```

$$K := (\,2\,sqwv\,)@\left(t \to \frac{1}{2}\,t\right)$$

Now K is the squarewave function we want.

```
diffeq := diff(y(t),t,t) + 5*y(t) = K(t);
```

$$diffeq := \left(\frac{\partial^2}{\partial t^2}\,y(\,t\,)\right) + 5\,y(\,t\,) = 2\,sqwv\left(\frac{1}{2}\,t\right)$$

```
f := unapply(rhs(dsolve({diffeq,y(0)=0,D(y)(0)=0},y(t))),t);
```

$$f := t \to -\frac{2}{5}\sqrt{5}\int_0^t sqwv\left(\frac{1}{2}\,u\right)\,sin\left(\sqrt{5}\,u\right)\,du\,cos\left(\sqrt{5}\,t\right) + \frac{2}{5}\sqrt{5}\int_0^t sqwv\left(\frac{1}{2}\,u\right)\,cos\left(\sqrt{5}\,u\right)\,du\,sin\left(\sqrt{5}\,t\right)$$

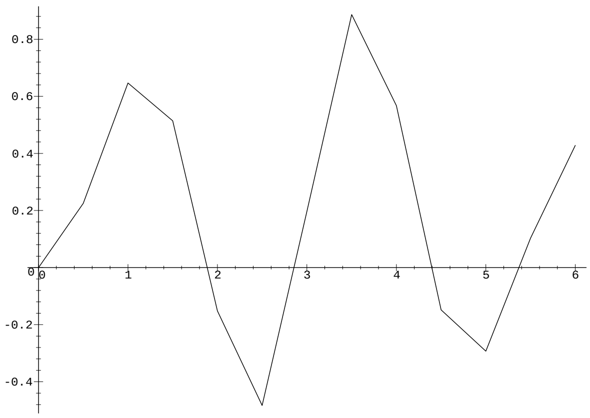
Note that dsolve returns a function defined by an integral as the solution. These are the integrals that arise in the variation of parameter method of solution. So the evaluation of f(3.) even is very expensive, about 7 seconds.

```
time();
evalf(f(3.));
time();
```

```
                                   470.000
                                  .1961199677
                                   477.000
```

So to get a facsimile of the graph of f over 0 to 10 takes a fair amount of time, and the picture isn't that good.

```
graph := 0 , evalf(f(0));
for t from .5 by .5 to 10 do graph := graph,t,evalf(f(t)) od:
plot([graph],title=`A Crude attempt`);
```



That doesn't work very well. So we break the problem into a series of initial value problems on the successive intervals 0..1, 1..2, ... , 9,10. The idea is that the right hand side becomes a simple constant function (either 0 or 2) and so the integrations above can be carried out symbolically. The initial values for each successive IVP are the values of the solution of the IVP on the preceding interval at the right hand endpoints of that interval. So we sort of 'walk' out a solution. At the end, we piece all of the solutions together.

First. Solve the IVP $y'' + 5y = 2$, $y(0) = 0$,$y'(0) = 0$ on the interval 0,1. call the soln f1

```
diffeq1 := diff(y(t),t,t) + 5*y(t) = 2;
f1 := unapply(rhs(dsolve({diffeq1,y(0)=0,D(y)(0)=0},y(t))),t);
df1 := unapply(diff(f1(t),t),t);
```

$$diffeq1 := \left(\frac{\partial^2}{\partial t^2}\, y(\,t\,)\right) + 5\, y(\,t\,) = 2$$

$$f1 := t \rightarrow \frac{2}{5} - \frac{2}{5}\, cos\left(\sqrt{5}\, t\right)$$

$$df1 := t \rightarrow \frac{2}{5}\, sin\left(\sqrt{5}\, t\right)\, \sqrt{5}$$

Then, for each i from 2 to 10, let fi be the solution of the IVP $y'' + 5y = SQW(t, 50, 2)$, $y(i-1) = fi(i-1)$, $y'(i-1) = fi'(i-1)$, on the interval $i-1..i$.

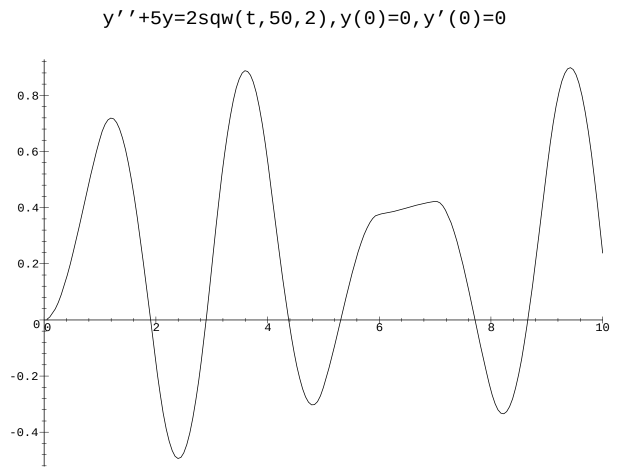
```
for i from 2 to 10 do
diffeq.i := diff(y(t),t,t) + 5*y(t) = 2*(i-floor(i/2)*2);
f.i := unapply(rhs(dsolve({diffeq.i,y(i-1)=evalf(f.(i-1)(i-1)),D(y)(i-1)=evalf(df.(i-1)(i-1))},
df.i := unapply(diff(f.i(t),t),t) od:
```

Now, construct the soln to the IVP $y'' + 5y = 2SQW(t, 50, 2)$, $y(0) = 0$, $y'(0) = 0$ by piecing together the solns we just found. Then graph them over the interval 0 to 10.

```
f :=  proc(t)
```

```
f.(ceil(t))(t) end;
f := proc(t) f.(ceil(t))(t) end
```

```
plot(f,0..10,title=`y''+5y=2sqw(t,50,2),y(0)=0,y'(0)=0`);
```



This is much better. But what if we wanted to know what the graph looked over the interval 5000 to 5010? We would need to know the values of f and f' at 5000.

6.13 The Laplace Transform

The Laplace Transform is a linear operator which is used to transform a linear differential equation involving an unknown function f(t) to an (linear) algebraic equation in the transform of f(t), F(s). The algebraic equation is solved for F(s) and the inverse Laplace transform is applied to obtain the function f(t). Amazing.

```
restart;
alias(H=Heaviside);
readlib(laplace);
```

$$I, H$$

```
proc(expr,t,s) ... end
```

Let's calculate a few transforms by hand. For example take $f(t) = \sin(wt)$

```
F := Int(sin(w*t)*exp(-s*t),t=0..R);
```

$$F := \int_0^R sin(\,w\,t\,)\,e^{(\,-s\,t\,)}\,dt$$

Load in the student package and do an integration by parts twice.

```
with (student):
eq1 :=  F = intparts(F,sin(w*t));
```

$$eq1 := \int_0^R sin(\,w\,t\,)\,e^{(-s\,t)}\,dt = -\frac{sin(\,w\,R\,)\,e^{(-s\,R)}}{s} - \int_0^R -\frac{cos(\,w\,t\,)\,w\,e^{(-s\,t)}}{s}\,dt$$

```
eq2  := lhs(eq1) = op(1,rhs(eq1)) + intparts(op(2,rhs(eq1)),cos(w*t));
```

$$eq2 := \int_0^R sin(\,w\,t\,)\,e^{(-s\,t)}\,dt = \\ -\frac{sin(\,w\,R\,)\,e^{(-s\,R)}}{s} - \frac{cos(\,w\,R\,)\,e^{(-s\,R)}\,w}{s^2} + \frac{w}{s^2} + \int_0^R -\frac{sin(\,w\,t\,)\,w^2\,e^{(-s\,t)}}{s^2}\,dt$$

We can look at this and see how to solve for `int(sin(wt)exp(-st),t=0..R)`.

```
eq3 := lhs(eq2)-op(4,rhs(eq2)) = rhs(eq2)-op(4,rhs(eq2)) ;
```

$$eq3 := \int_0^R sin(\,w\,t\,)\,e^{(-s\,t)}\,dt - \int_0^R -\frac{sin(\,w\,t\,)\,w^2\,e^{(-s\,t)}}{s^2}\,dt = \\ -\frac{sin(\,w\,R\,)\,e^{(-s\,R)}}{s} - \frac{cos(\,w\,R\,)\,e^{(-s\,R)}\,w}{s^2} + \frac{w}{s^2}$$

```
eq4 := Int(sin(w*t)*exp(-s*t),t=0..R) = rhs(eq3)/(1+w^2/s^2);
```

$$eq4 := \int_0^R sin(\,w\,t\,)\,e^{(-s\,t)}\,dt = \frac{-\frac{sin(\,w\,R\,)\,e^{(-s\,R)}}{s} - \frac{cos(\,w\,R\,)\,e^{(-s\,R)}\,w}{s^2} + \frac{w}{s^2}}{1+\frac{w^2}{s^2}}$$

```
simplify(eq4);
```

$$\int_0^R sin(\,w\,t\,)\,e^{(-s\,t)}\,dt = -\frac{sin(\,w\,R\,)\,e^{(-s\,R)}\,s + cos(\,w\,R\,)\,e^{(-s\,R)}\,w - w}{s^2+w^2}$$

Now taking the limit as R goes to infinity. we get the laplace transform is $w/(w^2+s^2)$, assuming $s > 0$. Checking this out with the Maple word laplace

```
laplace(sin(w*t),t,s);
```

$$\frac{w}{s^2+w^2}$$

Now lets use the Laplace transform to solve an initial value problem. Say take the second order equation $y'' + ay' + y = cos(wt)$;

```
f := 'f':
```

Set up the differential equation.

```
diffeq := diff(f(t),t,t)+a*diff(f(t),t)+f(t)=cos(w*t);
```

Apply the laplace transform to the equation.

```
slap:=laplace(diffeq,t,s);
```

Solve the resulting linear algebraic equation for `laplace(f(t),t,s)`.

```
sol1 :=solve(slap,laplace(f(t),t,s));
```

Apply the inverse laplace transform and make it into a function of t with the damping coefficient a and input frequency w.

```
g := unapply(invlaplace(sol1,s,t),(a,w,t)):
```

$$diffeq := \left(\frac{\partial^2}{\partial t^2} f(t)\right) + a\left(\frac{\partial}{\partial t} f(t)\right) + f(t) = cos(w\,t)$$

$$slap := (\,laplace(\,f(t), t, s)\,s - f(0)\,)\,s - D(\,f\,)(0) + a\,(\,laplace(\,f(t), t, s)\,s - f(0)\,) + laplace(\,f(t), t, s) = \frac{s}{s^2 + w^2}$$

$$sol1 := -\frac{-s\,f(0) - D(\,f\,)(0) - a\,f(0) - \frac{s}{s^2 + w^2}}{s^2 + a\,s + 1}$$

Now use these initial values and play with the parameters.

```
k := subs({f(0)=0,D(f)(0)=0},op(g)):
plots[animate](k(a,1,t),t=0..16*Pi,a=0..1.5,color=black,frames=4);
```

6.13.1 Problems

Problem 1. Work out the Laplace transform of x^2 by hand. Check your work with Maple.

Problem 2. Use the method of Laplace Transforms to solve the initial value problem $12y'' + 5y' - 2y = 0$, $y(0) = 3$, $y'(0) = 1$. Then find $y(2)$. Also find when $y(x) = 10$.

Problem 3. Use the method of Laplace Transforms to solve the initial value problem $12y'' + 5y' - 2y = x^2 + x - sin(x)$, $y(0) = 3$, $y'(0) = 1$. Then find $y(2)$. Also find when $y(x) = 10$.

6.14 Back to the Spring Problem

We can use the Laplace transform to solve the 2nd order equation $y''+5*y = SQW(t, 50, 2) = K(t)$, $y(0) = 0$, $y'(0) = 0$, in the interval 0 to 10. (Remember it?). The square wave from 0 to 10 can be written in terms of the Heaviside function.

```
restart;
alias(H=Heaviside);
readlib(laplace);
```

$$I, H$$

```
proc(expr,t,s) ... end
```

```
K := 2*sum(H@(t->(t-2*i+1))-H@(t->(t-2*i)),i=1..5);
```

$$K := 2\,H@(\,t \to t-1\,) - 2\,H@(\,t \to t-2\,) + 2\,H@(\,t \to t-3\,) - 2\,H@(\,t \to t-4\,) + 2\,H@(\,t \to t-5\,) - 2\,H@(\,t \to t-6\,) + 2\,H@(\,t \to t-7\,) - 2\,H@(\,t \to t-8\,) + 2\,H@(\,t \to t-9\,) - 2\,H@(\,t \to t-10\,)$$

Now Set up the differential equation.

```
diffeq := diff(f(t),t,t)+k*f(t)=K(t);
```

$$diffeq := \left(\frac{\partial^2}{\partial t^2} f(t)\right) + k\,f(t) = 2\,H(\,t-1\,) - 2\,H(\,t-2\,) + 2\,H(\,t-3\,) - 2\,H(\,t-4\,)$$

$$+ 2\,H(t-5) - 2\,H(t-6) + 2\,H(t-7) - 2\,H(t-8) + 2\,H(t-9) - 2\,H(t-10)$$

Apply the laplace transform to the equation.

```
slap:=laplace(diffeq,t,s);
```

$$slap := (\mathit{laplace}(f(t),t,s)\,s - f(0))\,s - D(f)(0) + k\,\mathit{laplace}(f(t),t,s) = 2\,\frac{e^{(-s)}}{s} - 2\,\frac{e^{(-2s)}}{s} + 2\,\frac{e^{(-3s)}}{s} - 2\,\frac{e^{(-4s)}}{s} + 2\,\frac{e^{(-5s)}}{s} - 2\,\frac{e^{(-6s)}}{s} + 2\,\frac{e^{(-7s)}}{s} - 2\,\frac{e^{(-8s)}}{s} + 2\,\frac{e^{(-9s)}}{s} - 2\,\frac{e^{(-10s)}}{s}$$

Solve the resulting linear algebraic equation for `laplace(f(t),t,s)`.

```
sol1 :=solve(slap,laplace(f(t),t,s));
sol2 := simplify(subs({f(0)=0,D(f)(0)=0},sol1));
```

$$sol1 := -\left(-s\,f(0) - D(f)(0) - 2\,\frac{e^{(-s)}}{s} + 2\,\frac{e^{(-2s)}}{s} - 2\,\frac{e^{(-3s)}}{s} + 2\,\frac{e^{(-4s)}}{s} - 2\,\frac{e^{(-5s)}}{s} + 2\,\frac{e^{(-6s)}}{s} - 2\,\frac{e^{(-7s)}}{s} + 2\,\frac{e^{(-8s)}}{s} - 2\,\frac{e^{(-9s)}}{s} + 2\,\frac{e^{(-10s)}}{s}\right) \Big/ (s^2 + k)$$

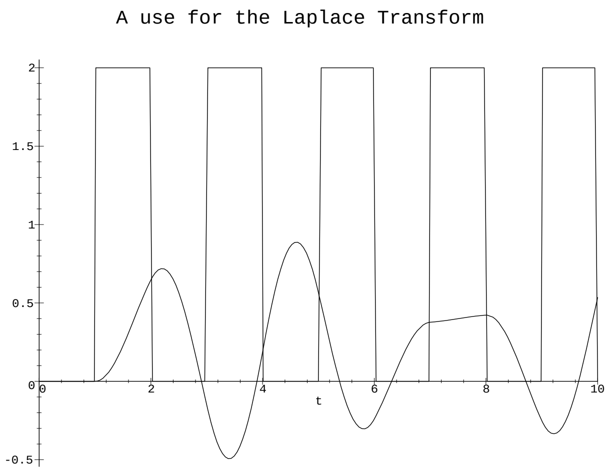
$$sol2 := -2(-e^{(-s)} + e^{(-2s)} - e^{(-3s)} + e^{(-4s)} - e^{(-5s)} + e^{(-6s)} - e^{(-7s)} + e^{(-8s)} - e^{(-9s)} + e^{(-10s)}) \Big/ (s\,(s^2+k))$$

Apply the inverse laplace transform and turn it into a function of time and the spring constant.

```
g := unapply(invlaplace(sol2,s,t),(t,k));
```

$$g := (t,k) \to 2\,H(t-1)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-1)\right)}{k}\right) - 2\,H(t-2)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-2)\right)}{k}\right) + 2\,H(t-3)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-3)\right)}{k}\right) - 2\,H(t-4)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-4)\right)}{k}\right) + 2\,H(t-5)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-5)\right)}{k}\right) - 2\,H(t-6)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-6)\right)}{k}\right) + 2\,H(t-7)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-7)\right)}{k}\right) - 2\,H(t-8)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-8)\right)}{k}\right) + 2\,H(t-9)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-9)\right)}{k}\right) - 2\,H(t-10)\left(\frac{1}{k} - \frac{\cos\left(\sqrt{k}\,(t-10)\right)}{k}\right)$$

```
plot({g(t,5),K(t)},t=0..10,color=black,title=`A use for the Laplace Transform`);
```



Voila!

6.14.1 Problem

Problem Make a movie out of the motion of the spring over the interval 0..10 as the spring constant increases from 1 to 5.

```
plots[animate]({g(t,k),K(t)},t=0..10,k=1..5,color=black,scaling=constrained);
```

This method is superior to our previous attempts, but we still don't have a good way to evaluate the solution out around t = 5000.

One more time

We can use the Laplace transform again to solve the 2nd order equation we have solved before. (Does this sound familiar?) $y'' + 5 * y = SQW(t, 50, 2)$, $y(0) = 0$,$y'(0) = 0$.

```
restart;
readlib(laplace):
```

But this time, instead of writing the square wave in terms of the Heaviside function, we approximate it with a linear combination of the functions $\sin(nt)$, for $n = 1, 3, 5, ...$

```
sq := 0:
for i from 1 to 7 do sq := sq + int(sin((2*i-1)*t),t=0..Pi)*sin((2*i-1)*t) od:
```

Also define k as a constant function

```
spring := t->k;
```

This gives a pretty darn good approximation of a square wave.

```
sq := unapply(sq,t);
```

$$sq := t \to 2\,sin(\,t\,) + \frac{2}{3}\,sin(\,3\,t\,) + \frac{2}{5}\,sin(\,5\,t\,) + \frac{2}{7}\,sin(\,7\,t\,) + \frac{2}{9}\,sin(\,9\,t\,) + \frac{2}{11}\,sin(\,11\,t\,)$$

$$+ \frac{2}{13} sin(13t)$$

But first we have to scale it so that it approximates the right hand side of our system.

```
f := 'f':
```

Now Set up the differential equation.

```
diffeq := diff(f(t),t,t)+spring(t)*f(t)=2/Pi*sq(t*Pi)+1:
```

Apply the laplace transform to the equation.

```
slap:=laplace(diffeq,t,s):
```

Solve the resulting linear algebraic equation for `laplace(f(t),t,s)`.

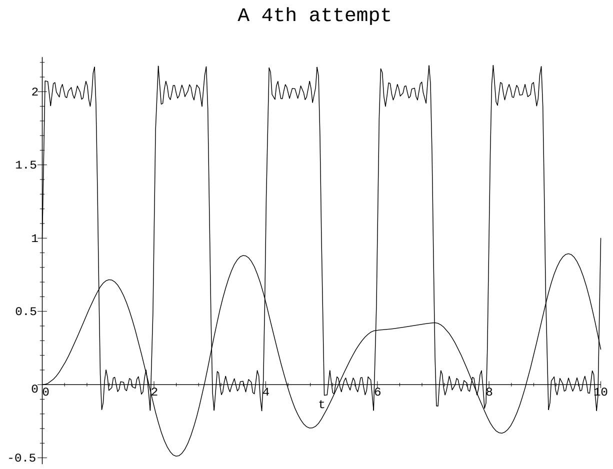
```
sol1 :=solve(slap,laplace(f(t),t,s)):

sol2 := simplify(subs({f(0)=0,D(f)(0)=0},sol1)):
```

Apply the inverse laplace transform and turn it into a function of time and the spring constant.

```
g := unapply(invlaplace(sol2,s,t),(t,k)):
```

First we check the solution over the interval [0, 10] with spring constant = 5.

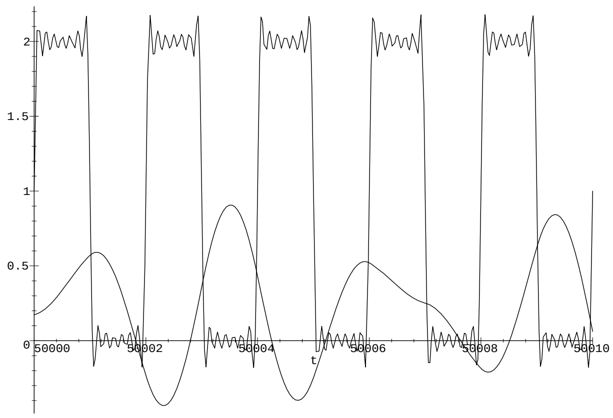
```
plot({g(t,5),2/Pi*sq(t*Pi)+1},t=0..10,color=black,title=`A 4th attempt`);
```



But, with this solution, we can check the motion after a long time has passed, say 50000 seconds.

```
plot({g(t,5),2/Pi*sq(t*Pi)+1},t=50000..50010,color=black,title=`Motion after a long time`);
```

Motion after a long time



It hasn't really changed that much. However, it is important to realize here that the right hand side is not the original square wave, just an approximation of it.

6.15 Systems 1: A system of salt tanks.

The Laplace transform is very useful in solving systems of differential equations. Here is an example.

Problem. Suppose we have three salt tanks holding 50, 30, and 20 gallons of brine respectively. Tank 1, called x1 below, has brine coming into it from two pipes. One pipe supplies brine at 1/2 gallon per minute with a concentration of g(t) lbs/gallon of salt (g(t) to be decided later). The second pipe pumps brine from tank3 into tank1 at 1/2 gallon per minute. Brine is pumped out of tank1 into tank2 at the rate of 1 gallon per minute, so the number of gallons of brine in tank1 remains at 50. Also brine is pumped out of tank2 into tank3 at the rate of 1 gallon per minute and pumped out of tank3 back to tank1 at 1/2 gallon/minute and to the drain at 1/2 gallon per minute. Find the functions giving the amount of salt in the tanks at time t.

Solution. We are given three differential equations in the three unknown functions xi(t), the number of lbs of salt in the tank xi at time t.

```
restart;
tank1 := diff(x1(t),t) = 1/2*g(t) +1/2*(x3(t)/20)-x1(t)/50;
tank2 := diff(x2(t),t) = x1(t)/50 - x2(t)/30;
tank3 := diff(x3(t),t) = x2(t)/30 - x3(t)/20;
```

$$tank1 := \frac{\partial}{\partial t}\, x1(\,t\,) = \frac{1}{2}\, g(\,t\,) + \frac{1}{40}\, x3(\,t\,) - \frac{1}{50}\, x1(\,t\,)$$

$$tank2 := \frac{\partial}{\partial t}\, x2(\,t\,) = \frac{1}{50}\, x1(\,t\,) - \frac{1}{30}\, x2(\,t\,)$$

$$tank3 := \frac{\partial}{\partial t}\,x3(\,t\,) = \frac{1}{30}\,x2(\,t\,) - \frac{1}{20}\,x3(\,t\,)$$

Now impose the initial conditions, say tanks x1, x2, and x3 have 10, 6 and 4 lbs of salt respectively at time 0. Also, choose the concentration of the incoming brine: let's make it fluctuate between 0 and 2 lbs/gallon with fairly slow oscillation.

```
g := t -> 1+cos(t/20);
inits := x1(0)=10,x2(0)=6,x3(0)=4;
fncs := {x1(t),x2(t),x3(t)};
```

$$g := t \rightarrow 1 + cos\left(\frac{1}{20}\,t\right)$$

$$inits := x1(\,0\,) = 10, x2(\,0\,) = 6, x3(\,0\,) = 4$$

$$fncs := \{\,x1(\,t\,), x2(\,t\,), x3(\,t\,)\,\}$$

Transform the system to a system of linear equations and solve the system.

```
slap :=solve(laplace({tank1,tank2,tank3},t,s),laplace(fncs,t,s));
```

$$\begin{aligned} slap := \{&laplace(\,x2(\,t\,), t, s\,) = 30(800\,s^2 + 800\,x1(\,0\,)\,s^3 + 2\,x1(\,0\,)\,s + 16000\,s^3 \\ &+ 56000\,x2(\,0\,)\,s^4 + 140\,x2(\,0\,)\,s^2 + 2800\,x2(\,0\,)\,s^3 + 2\,x2(\,0\,)\,s \\ &+ 400\,x3(\,0\,)\,s^3 + x3(\,0\,)\,s + 800000\,x2(\,0\,)\,s^5 + 16000\,x1(\,0\,)\,s^4 \\ &+ 40\,x1(\,0\,)\,s^2 + 1 + 20\,s)\Big/(\,(\,60000\,s^3 + 6200\,s^2 + 200\,s + 1\,)\,s \\ &(\,400\,s^2 + 1\,)\,), laplace(\,x1(\,t\,), t, s\,) = 50(1400\,s^2 + 2000\,x1(\,0\,)\,s^3 \\ &+ 2\,x1(\,0\,)\,s + 40000\,s^3 + 400\,x2(\,0\,)\,s^3 + 480000\,s^4 + x2(\,0\,)\,s \\ &+ 400\,x3(\,0\,)\,s^3 + 12000\,x3(\,0\,)\,s^4 + 30\,x3(\,0\,)\,s^2 + x3(\,0\,)\,s \\ &+ 40000\,x1(\,0\,)\,s^4 + 100\,x1(\,0\,)\,s^2 + 480000\,x1(\,0\,)\,s^5 + 1 + 50\,s)\Big/(\\ &(\,60000\,s^3 + 6200\,s^2 + 200\,s + 1\,)\,s\,(\,400\,s^2 + 1\,)\,), laplace(\,x3(\,t\,), t, s\,) = 20 \\ &(1200000\,x3(\,0\,)\,s^5 + 3800\,x3(\,0\,)\,s^3 + 64000\,x3(\,0\,)\,s^4 + 160\,x3(\,0\,)\,s^2 \\ &+ 2\,x3(\,0\,)\,s + 40000\,x2(\,0\,)\,s^4 + 100\,x2(\,0\,)\,s^2 + 800\,x2(\,0\,)\,s^3 + 2\,x2(\,0\,)\,s \\ &+ 800\,s^2 + 1 + 800\,x1(\,0\,)\,s^3 + 2\,x1(\,0\,)\,s)\Big/(\\ &(\,60000\,s^3 + 6200\,s^2 + 200\,s + 1\,)\,s\,(\,400\,s^2 + 1\,)\,)\} \end{aligned}$$

Now substitute in the initial conditions.

```
slap:= subs({inits},slap);
```

$$\begin{aligned} slap := \Big\{&laplace(\,x2(\,t\,), t, s\,) = \\ &30\,\frac{2040\,s^2 + 42400\,s^3 + 56\,s + 496000\,s^4 + 4800000\,s^5 + 1}{(\,60000\,s^3 + 6200\,s^2 + 200\,s + 1\,)\,s\,(\,400\,s^2 + 1\,)}, \\ &laplace(\,x1(\,t\,), t, s\,) = \end{aligned}$$

$$50\,\frac{2520\,s^2+64000\,s^3+80\,s+928000\,s^4+4800000\,s^5+1}{(\,60000\,s^3+6200\,s^2+200\,s+1\,)\,s\,(\,400\,s^2+1\,)},$$
$$laplace(\,x3(\,t\,),t,s\,)=$$
$$\left.20\,\frac{4800000\,s^5+28000\,s^3+496000\,s^4+2040\,s^2+40\,s+1}{(\,60000\,s^3+6200\,s^2+200\,s+1\,)\,s\,(\,400\,s^2+1\,)}\right\}$$

Now apply the inverse laplace transform

```
sol := invlaplace(slap,s,t);
```

$$sol:=\Bigg\{x2(\,t\,)=30+\left(\sum_{\_r=\%1}\frac{(\,-1890960-63026400\,\_r-580320000\,\_r^2\,)\,e^{(\,\_r\,t\,)}}{77940000\,\_r^2+5369200\,\_r+86600}\right)$$
$$+\frac{1020}{433}\,sin\left(\frac{1}{20}\,t\right)-\frac{720}{433}\,cos\left(\frac{1}{20}\,t\right),\,x1(\,t\,)=50$$
$$+\left(\sum_{\_r=\%1}\frac{(\,-2668000-102872000\,\_r-1120200000\,\_r^2\,)\,e^{(\,\_r\,t\,)}}{77940000\,\_r^2+5369200\,\_r+86600}\right)$$
$$+\frac{3500}{433}\,sin\left(\frac{1}{20}\,t\right)+\frac{1350}{433}\,cos\left(\frac{1}{20}\,t\right),\,x3(\,t\,)=20$$
$$+\left(\sum_{\_r=\%1}\frac{(\,-1387600-39657600\,\_r-380880000\,\_r^2\,)\,e^{(\,\_r\,t\,)}}{77940000\,\_r^2+5369200\,\_r+86600}\right)$$
$$\left.+\frac{100}{433}\,sin\left(\frac{1}{20}\,t\right)-\frac{580}{433}\,cos\left(\frac{1}{20}\,t\right)\right\}$$
$$\%1:=RootOf(\,60000\,\_Z^3+6200\,\_Z^2+200\,\_Z+1\,)$$

assign and convert the solutions to functions.

```
assign(evalf(sol));
for i from 1 to 3 do
x.i := unapply(x.i(t),t)
od;
```

$$x1:=t\rightarrow 50.-6.454938419\,e^{(-.04862778518\,t\,)}\,cos(\,.01943130303\,t\,)$$
$$-2.103537842\,e^{(-.04862778518\,t\,)}\,sin(\,.01943130303\,t\,)$$
$$-36.66284449\,e^{(-.0060777629681348\,t\,)}+8.083140878\,sin(\,.05000000000\,t\,)$$
$$+3.117782910\,cos(\,.05000000000\,t\,)$$

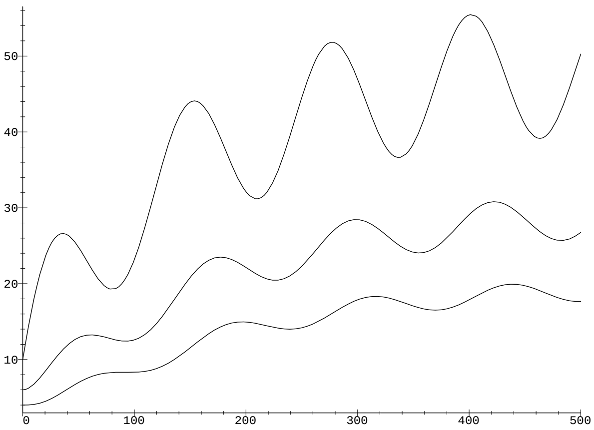
$$x2:=t\rightarrow 30.+4.565827835\,e^{(-.04862778518\,t\,)}\,cos(\,.01943130303\,t\,)$$
$$-3.050075144\,e^{(-.04862778518\,t\,)}\,sin(\,.01943130303\,t\,)$$
$$-26.90301028\,e^{(-.0060777629681348\,t\,)}+2.355658199\,sin(\,.05000000000\,t\,)$$
$$-1.662817552\,cos(\,.05000000000\,t\,)$$

$$x3:=t\rightarrow 20.+5.756644367\,e^{(-.04862778518\,t\,)}\,cos(\,.01943130303\,t\,)$$
$$+7.425899758\,e^{(-.04862778518\,t\,)}\,sin(\,.01943130303\,t\,)$$

$$- 20.41715245\, e^{(-.006077762968134 8t)} + .2309468822\, sin(\, .05000000000\, t\,)$$
$$- 1.339491917\, cos(\, .05000000000\, t\,)$$

Then plot the functions over the interval 0 to 500.

```
plot({x1,x2,x3},0..500,color=black);
```



6.15.1 Problems

Problem 1. What happens to the solution if we change the amount of salt in tank 1 to be 0 lbs at time 0? Will the long term trend of the solution be changed?

Problem 2. What happens to the solution if we change the input concentration $g(t)$ to a constant 1 lb/gal.? Will the long term trend of the solution be changed?

Problem 3. Change the original problem by running the drainline from tank 3 back to tank 2 and then having a drain out of tank 2. Solve and compare with the solution to the original problem.

6.16 Nonlinear Systems

Predator-Prey equations: These equations have been used to model the populations of two interacting species, where one of the species is a prey whose growth is decreased as the other species (the predator) increases. These are nonlinear first order equations, and cannot be reduced to a single second order linear equation as we did in the case of a linear first order system. However, we can study the system by use of the word `phaseportrait`.

First set up the equations.

```
prey := (a,b) -> (a- b*y)*x;
pred := (c,d) -> (c*x - d)*y;
```

$$prey := (\,a, b\,) \to (\,a - b\,y\,)\,x$$

$$pred := (c, d) \rightarrow (c\,x - d)\,y$$

Now load up DEtools.

```
with(DEtools);
```

$[\,DEplot, DEplot1, DEplot2, Dchangevar, PDEplot, dfieldplot, phaseportrait\,]$

```
phaseportrait([prey(.9,.1),pred(.1,.7)],[x,y],0..5,{[0,20,4]},stepsize=.2,arrows=THIN);
```

The critical points of the predator-prey system are the points (x,y) where both rates of change are 0. These can be seen to be at the origin (0,0) and (a/b, c/d). Describe the behavior of the solution curves near these critical points.

What happens to the model if we change the prey equation to be logistic in the absence of the predator?

```
prey := (a,e,b) -> (a- e*x- b*y)*x;
pred := (c,d) -> (c*x - d)*y;
```

$$prey := (a, e, b) \rightarrow (a - e\,x - b\,y)\,x$$

$$pred := (c, d) \rightarrow (c\,x - d)\,y$$

```
phaseportrait([prey(.9,.04,.1),pred(.1,.7)],[x,y],0..5,{[0,20,4]},stepsize=.2,arrows=THIN);
```