notes:

Taylor's Theorem for the case n =1

1 *Theorem: Suppose f has a continuous 2nd derivative on the interval [a,b]. Let $\mathrm{P}(x) = \mathrm{f}(a) + f'(a)(x - a)$ be the tangent line to f at x=a and let $\mathrm{R}(x) = \mathrm{f}(x) - \mathrm{P}(x)$ be the remainder. Then Taylor says that $\mathrm{R}(b) = \frac{f''(\alpha)\,(b-a)^2}{2}$ for some α in the interval from a to b, and more generally, for each c and x in [a,b], $\mathrm{R}(x) = \frac{f''(\alpha)\,(x-c)^2}{2}$ for some α between c and x.*

The proof depends on a fact about integrals: If f and g are continuous on an interval with endpoints a and b, then

$$\int_a^b \mathrm{f}(t)\,\mathrm{g}(t)\,dt = \mathrm{f}(\tau)\ \int_a^b \mathrm{g}(t)\,dt$$

for some τ between a and b.

Proof: (Modelled on a proof in Buck's *Advanced Calculus*, first edition.)

Define a function $\mathrm{g}(s,\,t) = \mathrm{f}(s) + f'(s)(t - s)$, and write the remainder R(b) in terms of g.

$$\mathrm{R}(b) = \mathrm{g}(b,\,b) - \mathrm{g}(a,\,b)$$

But now the difference on the right hand side is the integral from a to b of the partial derivative of g with respect to the first variable.

$$\mathrm{g}(b,\,b) - \mathrm{g}(a,\,b) = \int_a^b g_1(s,\,b)\,ds$$

Differentiating g with respect to s, we get

$$g_1(s,\,b) = f'(s) + f''(s)(b - s) + f'(s)(-1)$$

and the right hand side reduces to $f''(s)(b - s)$. Now use the fact about integrals:

$$\int_a^b f''(s)(b - s)\,ds = f''(\alpha)\ \int_a^b (b - s)^2\,ds$$

for some α in [a,b]. Integrating the right hand side, we get

$$f''(\alpha)\ \int_a^b (b - s)^2\,ds = \frac{f''(\alpha)\,(b-a)^2}{2}.$$

This establishes the first statement. The more general statement follows if we simply repeat the same argument with a and b replaced by c and x throughout.

Assignment:

The hyperbolic cosine, cosh, is defined as the average of exp(x) and exp(-x). Get the Taylor series for cosh about a=0. How many terms do you need to guarantee 7 digits of accuracy (rounded) as x ranges over the interval 0 to 2? Do the calculations with 14 digits of precision to avoid roundoff error. Write a proc 'mycosh' which will take any x in 0 to 2, evaluate the Taylor polynomial you have determined above and return the value rounded to 7 digits. Make a table comparing the values of mycosh with Maples cosh function.