MA 214 Calculus IV (Spring 2016)
Section 2

Homework Assignment 11
Solutions

1. Boyce and DiPrima, Section 5.1, p. 253, Problem 24 and Problem 26.

   Solution: Problem 24. It is easy to see that

$$\begin{aligned}(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right]x^n.\end{aligned}$$

   Problem 27. By shifting indices at the second step, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} na_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= a_1 + \sum_{n=1}^{\infty}[(n+1)na_{n+1} + a_{n-1}]x^n.\end{aligned}$$

2. Boyce and DiPrima, Section 5.1, p. 254, Problem 28.

   Solution: By shifting the index of the first summand, we obtain

$$\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\big((n+1)a_{n+1} + 2a_n\big)x^n = 0$$

   for each x. Hence we obtain the recurrence relation

$$a_{n+1} = \frac{-2}{n+1}a_n \qquad \text{for } n = 0, 1, 2, \cdots,$$

which implies

$$a_1 = -2a_0, \quad a_2 = \frac{-2}{2}a_1 = \frac{(-2)^2}{2}a_0, \quad a_3 = \frac{-2}{3}a_2 = \frac{(-2)^3}{3!}a_0, \quad a_4 = \frac{-2}{4}a_3 = \frac{(-2)^4}{4!}a_0,$$

and

$$a_n = \frac{(-2)^n}{n!}a_0 \qquad \text{for } n = 0, 1, 2, \cdots.$$

It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= a_0 \left(1 + (-2)x + \frac{(-2)^2}{2!}x^2 + \frac{(-2)^3}{3!}x^3 + \cdots + \frac{(-2)^n}{n!}x^n + \cdots\right) \\
&= a_0 \left(1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots + \frac{(-2x)^n}{n!} + \cdots\right) \\
&= a_0 \exp(-2x).
\end{aligned}$$

3. Boyce and DiPrima, Section 5.2, p. 263, Problem 1.

Solution: Let $L(y) = y'' - y$ and $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$, and

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n.$$

Hence we have

$$L(y) = \sum_{n=0}^{\infty} \big(c_{n+2}(n+2)(n+1) - c_n\big)x^n = 0 \qquad \text{for each } x \in (-\infty, \infty),$$

which implies the recurrence relation

$$c_{n+2}(n+2)(n+1) - c_n = 0 \qquad \text{or} \qquad c_{n+2} = \frac{1}{(n+2)(n+1)}c_n, \qquad (n = 0, 1, 2, \cdots).$$

For solution y_1, we put $c_0 = 1$ and $c_1 = 0$. Then $c_{2m+1} = 0$ for each $m = 0, 1, 2, \cdots$,

$$c_2 = \frac{1}{2 \cdot 1}c_0 = \frac{1}{2!}, \quad c_4 = \frac{1}{4 \cdot 3}c_2 = \frac{1}{4!}, \quad c_6 = \frac{1}{6 \cdot 5}c_4 = \frac{1}{6!},$$

and

$$c_{2m} = \frac{1}{2m \cdot (2m-1) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{(2m)!} \qquad \text{for } m = 0, 1, 2, \cdots.$$

Therefore the solution y_1 is given by the formula

$$y_1 = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots = \sum_{m=0}^{\infty} \frac{1}{(2m)!}x^{2m}.$$

For solution y_2, we put $c_0 = 0$ and $c_1 = 1$. Then $c_{2m} = 0$ for each $m = 0, 1, 2, \cdots$,

$$c_3 = \frac{1}{3 \cdot 2} c_1 = \frac{1}{3!}, \quad c_5 = \frac{1}{5 \cdot 4} c_3 = \frac{1}{5!}, \quad c_7 = \frac{1}{7 \cdot 6} c_5 = \frac{1}{7!},$$

and

$$c_{2m+1} = \frac{1}{(2m+1) \cdot 2m \cdot \cdots, \cdot 3 \cdot 2 \cdot 1} = \frac{1}{(2m+1)!} \qquad \text{for } m = 0, 1, 2, \cdots.$$

Therefore the solution y_2 is given by the formula

$$y_2 = x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \frac{1}{7!} x^7 + \cdots = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{2m+1}.$$

The general solution of the given equation is given by

$$y = \alpha_1 y_1 + \alpha_2 y_2, \qquad \text{where } \alpha_1 \text{ and } \alpha_2 \text{ are arbitrary constants.}$$

4. Boyce and DiPrima, Section 5.2, p. 263, Problem 7.

Solution: Let $L(y) = y'' + xy' + 2y$ and $y = \sum_{n=0}^{\infty} c_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \qquad xy' = \sum_{n=1}^{\infty} c_n n x^n,$$
$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n.$$

Hence we have

$$L(y) = \sum_{n=0}^{\infty} \big(c_{n+2}(n+2)(n+1) x^n + c_n n + 2c_n \big) x^n = 0 \qquad \text{for each } x \in (-\infty, \infty),$$

which implies the recurrence relation

$$c_{n+2}(n+2)(n+1) + c_n(n+2) = 0 \qquad \text{or} \qquad c_{n+2} = \frac{-1}{n+1} c_n, \qquad (n = 0, 1, 2, \cdots).$$

For solution y_1, we put $c_0 = 1$ and $c_1 = 0$. Then $c_{2m+1} = 0$ for each $m = 0, 1, 2, \cdots$,

$$c_2 = \frac{-1}{1} c_0 = -1, \quad c_4 = \frac{-1}{3} c_2 = \frac{(-1)^2}{3 \cdot 1}, \quad c_6 = \frac{-1}{5} c_4 = \frac{(-1)^3}{5 \cdot 3 \cdot 1},$$

and

$$
\begin{aligned}
c_{2m} &= \frac{(-1)^m}{(2m-1)\cdot(2m-3)\cdot\cdots\cdot 5\cdot 3\cdot 1} \\
&= \frac{(-1)^m}{(2m-1)\cdot(2m-3)\cdot\cdots\cdot 5\cdot 3\cdot 1}\cdot\frac{2m\cdot(2m-2)\cdot\cdots\cdot 4\cdot 2}{2m\cdot(2m-2)\cdot\cdots\cdot 4\cdot 2} \\
&= \frac{(-1)^m}{(2m-1)\cdot(2m-3)\cdot\cdots\cdot 5\cdot 3\cdot 1}\cdot\frac{2m\cdot 2(m-1)\cdot\cdots\cdot 2(2)\cdot 2(1)}{2m\cdot(2m-2)\cdot\cdots\cdot 4\cdot 2} \\
&= \frac{(-1)^m 2^m m!}{(2m)!} \qquad \text{for } m = 0, 1, 2, \cdots.
\end{aligned}
$$

Therefore the solution y_1 is given by the formula

$$
y_1 = 1 - x^2 + \frac{1}{3\cdot 1}x^4 - \frac{1}{5\cdot 3\cdot 1}x^6 + \cdots = \sum_{m=0}^{\infty}\frac{(-1)^m 2^m m!}{(2m)!}x^{2m}.
$$

For solution y_2, we put $c_0 = 0$ and $c_1 = 1$. Then $c_{2m} = 0$ for each $m = 0, 1, 2, \cdots$,

$$
c_3 = \frac{-1}{2}c_1 = \frac{-1}{2}, \quad c_5 = \frac{-1}{4}c_3 = \frac{(-1)^2}{4\cdot 2}, \quad c_7 = \frac{-1}{6}c_5 = \frac{(-1)^3}{6\cdot 4\cdot 2},
$$

and

$$
\begin{aligned}
c_{2m+1} &= \frac{(-1)^m}{2m\cdot(2m-2)\cdot\cdots\cdot 6\cdot 4\cdot 2} \\
&= \frac{(-1)^m}{2m\cdot 2(m-1)\cdot\cdots\cdot 2(3)\cdot 2(2)\cdot 2(1)} \\
&= \frac{(-1)^m}{2^m m!} \qquad \text{for } m = 0, 1, 2, \cdots.
\end{aligned}
$$

Therefore the solution y_2 is given by the formula

$$
y_1 = x - \frac{1}{2}x^2 + \frac{1}{4\cdot 2}x^4 - \frac{1}{6\cdot 4\cdot 2}x^6 + \cdots = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}x^{2m+1},
$$

where $0! := 1$. The general solution of the given equation is given by

$$
y = \alpha_1 y_1 + \alpha_2 y_2, \qquad \text{where } \alpha_1 \text{ and } \alpha_2 \text{ are arbitrary constants.}
$$

5. Boyce and DiPrima, Section 5.2, p. 263, Problem 8.

Solution: To find a power series solution of the form $\sum_{n=0}^{\infty} c_n(x-1)^n$, we put $t = x - 1$. Then $x = t + 1$ and, with t as the independent variable, the given equation becomes

$$
(t+1)\frac{d^2y}{dt^2} + \frac{dy}{dt} + (t+1)y = 0.
$$

Let $\dot{y} = dy/dt$, $\ddot{y} = d^2y/dt^2$, and $L(y) = (t+1)\ddot{y} + \dot{y} + (t+1)y$. Substituting $y = \sum_{n=0}^{\infty} c_n t^n$ into the equation $L(y) = 0$, we have

$$\begin{aligned}
L(y) &= (t+1)\sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} + \sum_{n=1}^{\infty} c_n n t^{n-1} + (t+1)\sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=2}^{\infty} c_n n(n-1)t^{n-1} + \sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} + \sum_{n=1}^{\infty} c_n n t^{n-1} + \sum_{n=0}^{\infty} c_n t^{n+1} + \sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=1}^{\infty} c_{n+1}(n+1)n t^n + \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n + \sum_{n=0}^{\infty} c_{n+1}(n+1)t^n + \sum_{n=1}^{\infty} c_{n-1}t^n + \sum_{n=0}^{\infty} c_n t^n \\
&= \sum_{n=1}^{\infty} \big(c_{n+2}(n+2)(n+1) + c_{n+1}(n+1)^2 + c_n + c_{n-1}\big) + 2c_2 + c_1 - c_0 \\
&= 0 \qquad \text{for each } t \in (-1,1).
\end{aligned}$$

Thus we obtain the recurrence relation

$$c_{n+2} = -\left(\frac{c_{n+1}(n+1)^2 + c_n + c_{n-1}}{(n+2)(n+1)}\right) \qquad \text{for } n = 1, 2, \cdots,$$

and the requirement

$$2c_2 + c_1 + c_0 = 0.$$

For solution y_1, we put $c_0 = 1$ and $c_1 = 0$. Then

$$c_2 = -\frac{1}{2}, \quad c_3 = -\frac{4c_2 + c_0}{3 \cdot 2} = \frac{1}{6}, \quad c_4 = -\frac{9c_3 + c_2}{4 \cdot 3} = -\frac{1}{12},$$

and we have

$$y_1 = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots.$$

For solution y_2, we put $c_0 = 0$ and $c_1 = 1$. Then

$$c_2 = -\frac{1}{2}, \quad c_3 = -\frac{4c_2 + c_1}{3 \cdot 2} = \frac{1}{6}, \quad c_4 = -\frac{9c_3 + c_2 + c_1}{4 \cdot 3} = -\frac{1}{6},$$

and we have

$$y_2 = x - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \cdots.$$

The general solution of the given equation is given by

$$y = \alpha_1 y_1 + \alpha_2 y_2, \qquad \text{where } \alpha_1 \text{ and } \alpha_2 \text{ are arbitrary constants.}$$

6. Boyce and DiPrima, Section 5.2, p. 263, Problem 9.

Solution: Let $L(y) = (1+x^2)y'' - 4xy' + 6y$. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the equation $L(y) = 0$, we have

$$
\begin{aligned}
&(1+x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 4x\sum_{n=1}^{\infty} c_n n x^{n-1} + 6\sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} c_n n(n-1)x^n - \sum_{n=1}^{\infty} 4c_n n x^n + 6\sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} c_n n(n-1)x^n - \sum_{n=0}^{\infty} 4c_n n x^n + 6\sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} \big(c_{n+2}(n+2)(n+1) + c_n(n(n-1) - 4n + 6)\big)x^n \\
&= \sum_{n=0}^{\infty} \big(c_{n+2}(n+2)(n+1) + c_n(n-2)(n-3)\big)x^n \\
&= 0 \qquad \text{for each } x \in (-1,1).
\end{aligned}
$$

Thus we obtain the recurrence relation

$$c_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)}c_n \qquad \text{for } n = 0, 1, 2, \cdots.$$

For solution y_1, we put $c_0 = 1$ and $c_1 = 0$. Then $c_{2m+1} = 0$ for each $m = 0, 1, 2, \cdots$,

$$c_2 = -\frac{(-2)(-3)}{2}c_0 = -3, \quad c_4 = 0,$$

which implies $c_{2m} = 0$ for $m \geq 2$. Therefore the power-series of y_1 terminates at $n = 2$, and we have

$$y_1 = 1 - 3x^2.$$

For solution y_2, we put $c_0 = 0$ and $c_1 = 1$. Then $c_{2m} = 0$ for each $m = 0, 1, 2, \cdots$,

$$c_3 = -\frac{(-1)(-2)}{3 \cdot 2}c_1 = -\frac{1}{3}, \quad c_5 = 0,$$

which implies $c_{2m+1} = 0$ for $m \geq 2$. Therefore the power-series of y_2 terminates at $n = 3$, and we have

$$y_2 = x - \frac{1}{3}x^3.$$

The general solution of the given equation is given by

$$y = \alpha_1 y_1 + \alpha_2 y_2, \qquad \text{where } \alpha_1 \text{ and } \alpha_2 \text{ are arbitrary constants.}$$

7. Boyce and DiPrima, Section 5.2, p. 263, Problem 10.

Solution: Let $L(y) = (4 - x^2)y'' + 2y$. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the equation $L(y) = 0$, we have

$$\begin{aligned}
&(4 - x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 2\sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} 4c_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} c_n n(n-1)x^n + \sum_{n=0}^{\infty} 2c_n x^n \\
&= \sum_{n=0}^{\infty} 4c_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} c_n n(n-1)x^n + \sum_{n=0}^{\infty} 2c_n x^n \\
&= \sum_{n=0}^{\infty} \big(4c_{n+2}(n+2)(n+1) - c_n(n(n-1)-2)\big)x^n \\
&= \sum_{n=0}^{\infty} \big(4c_{n+2}(n+2)(n+1) - c_n(n+1)(n-2)\big)x^n = 0 \qquad \text{for } x \in (-2, 2).
\end{aligned}$$

Thus we obtain the recurrence relation

$$c_{n+2} = \frac{(n-2)}{4(n+2)}c_n \qquad \text{for } n = 0, 1, 2, \cdots.$$

For solution y_1, we put $c_0 = 1$ and $c_1 = 0$. Then $c_{2m+1} = 0$ for each $m = 0, 1, 2, \cdots$,

$$c_2 = \frac{-2}{8}c_0 = -\frac{1}{4}, \quad c_4 = 0,$$

which implies $c_{2m} = 0$ for $m \geq 2$. Therefore the power-series of y_1 terminates at $n = 2$, and we have

$$y_1 = 1 - \frac{1}{4}x^2.$$

For solution y_2, we put $c_0 = 0$ and $c_1 = 1$. Then $c_{2m} = 0$ for each $m = 0, 1, 2, \cdots$,

$$c_3 = \frac{-1}{4 \cdot 3}c_1 = -\frac{1}{4 \cdot 3}, \quad c_5 = \frac{1}{4 \cdot 5}c_3 = -\frac{1}{4^2 \cdot 5 \cdot 3}, \quad c_7 = \frac{3}{4 \cdot 7}c_5 = -\frac{1}{4^3 \cdot 7 \cdot 5},$$

and

$$c_{2m+1} = -\frac{1}{4^m(2m+1)(2m-1)} \qquad \text{for } m = 0, 1, 2, \cdots.$$

Therefore the solution y_2 is given by the formula

$$y_2 = x - \frac{1}{4 \cdot 3}x^3 - \frac{1}{4^2 \cdot 5 \cdot 3}x^5 - \frac{1}{4^3 \cdot 7 \cdot 5}x^7 + \cdots = -\sum_{m=0}^{\infty} \frac{1}{4^m(2m+1)(2m-1)}x^{2m+1}.$$

The general solution of the given equation is given by

$$y = \alpha_1 y_1 + \alpha_2 y_2, \qquad \text{where } \alpha_1 \text{ and } \alpha_2 \text{ are arbitrary constants.}$$