0. Introduction

- **Let** k **be a field, char** $k = 0$**,**
- $X = \{x_1, ..., x_n\}$ **a set of variables over** k
- I_1**,** I_2 **homogeneous ideals of** $k[X]$
- $R_1 = k[X]/I_1$ **,** $R_2 = k[X]/I_2$ **,** $V(I_1) \subset \mathbb{P}_k^{n-1}$, $V(I_2) \subset \mathbb{P}_k^{n-1}$

Consider the exact sequence

$$0 \longrightarrow \mathbb{D} \longrightarrow S = R_1 \otimes_k R_2 \xrightarrow{\textbf{mult.}} k[X]/(I_1, I_2) \longrightarrow 0.$$

- $\mathbb{D}$ **diagonal ideal of** $S = R_1 \otimes_k R_2 \cong k[X, Y]/(I_1, I_2)$**, where** $Y = (y_i)$**.**

- **Let φ be the surjective map**

$$\textbf{Sym}(\mathbb{D}) \twoheadrightarrow \mathcal{R}(\mathbb{D})$$

- $\textbf{Sym}(\mathbb{D}) = \oplus_{i \geq 0} \textbf{Sym}_i(\mathbb{D})$ **is the symmetric algebra of** $\mathbb{D}$**,**

- $\mathcal{R}(\mathbb{D}) = \oplus_{i \geq 0} \mathbb{D}^i$ **is the Rees algebra of** $\mathbb{D}$

- $\textbf{ker}(\varphi)$ **is the** S**-torsion of** $\textbf{Sym}(\mathbb{D})$

- **In general, it is not easy to find the defining equations of Rees algebras.**

- $\varphi \otimes_S k$ **is the map**

$$\textbf{Sym}(\mathbb{D}) \otimes_S k = k[X] \twoheadrightarrow \mathcal{R}(\mathbb{D}) \otimes_S k = \mathcal{F}(\mathbb{D})$$

- $\mathcal{F}(\mathbb{D})$ **is the special fiber ring of** $\mathbb{D}$**.**

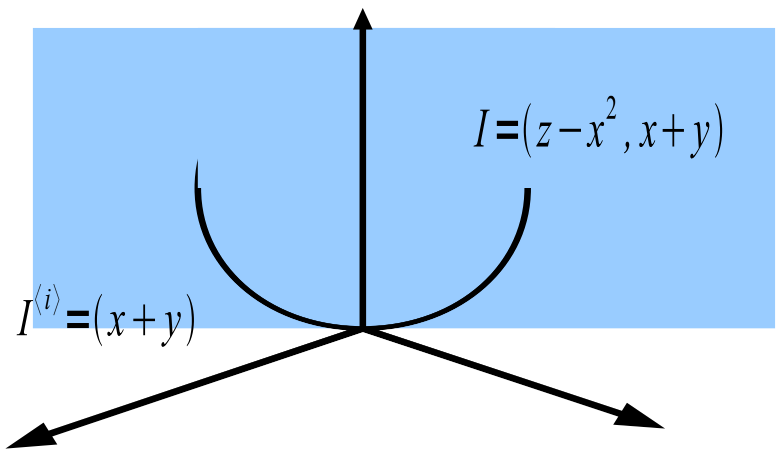
- $\mathcal{F}(\mathbb{D})$ **is the homogeneous coordinate ring of the embedded join variety** $\mathcal{J}(I_1, I_2) \subset \mathbb{P}_k^{n-1}$ **of** $V(I_1)$, $V(I_2)$.

- **When** $I = I_1 = I_2$, **the embedded join variety** $V(I^{<1>}) = \mathcal{J}(I, I)$ **is known as the first secant variety of** $V(I)$, **which is constructed by taking the closure of the secant lines of** $V(I)$.

- **We write the** s**-th secant variety of** I **as** $V(I^{<s>})$, **which is constructed as the embedded join variety of** $V(I^{<s-1>})$ **and** $V(I)$.

- **In general, we have** $V(I^{<s-1>}) \subseteq V(I^{<s>})$.

Example: $R = k[x, y, z]$



General question: When does the secant variety fill out the whole space?

- **If we have the secant variety that does not fill out space, we can define a map to reduce the dimension of the variety without losing the properties of the variety.**

• **We write $\mathbb{D}^{<s>}$ for the ideal of the diagonal in the product of $V(I^{<s-1>})$ and $V(I)$.**

• **The special fiber ring of $\mathbb{D}^{<s>}$, $\mathcal{F}(\mathbb{D}^{<s>})$, gives the homogeneous coordinate ring of s-th secant variety.**

• **We are interested in the determinantal ring $k[X]$, where $X = (x_{ij})$, an m by n matrix of variables over k or a m by m symmetric matrix of variables over k.**

• **We focus on the ideal generated by the 2 by 2 minors of the above two matrices.**

Known Fact:

Let k be a field of characteristic zero, and let $I = I_2(X)$ in $R = k[X]$. The defining ideal $I^{<s>}$ of the s-th secant variety of $V(I)$ is $I_{s+2}(X)$.

• **For $s+2>m$, we have $I^{<s>}=0$, i.e. $\mathcal{F}(\mathbb{D}^{<s>})=k[X]$, hence $\varphi\otimes_S k:\ k[X]\twoheadrightarrow\mathcal{F}(\mathbb{D}^{<s>})$ is an isomorphism.**

Proposition

If $\mathcal{F}(\mathbb{D}^{<s>})=k[X]$ then we have the $\mathcal{R}(\mathbb{D}^{<s+1>})=\mathrm{Sym}(\mathbb{D}^{<s+1>})$.

• **With the fact and the proposition, two questions are raised:**

Question 1: Is $\mathcal{R}(\mathbb{D}^{<s>})=$ Sym$(\mathbb{D}^{<s>})$? ,

i.e. what is the minimal s such that $\mathcal{R}(\mathbb{D}^{<s>})=$ Sym$(\mathbb{D}^{<s>})$?

Question 2: If the equality does not hold for s, can we find the defining equations of $\mathcal{R}(\mathbb{D}^{<s>})$?

What are properties of $\mathcal{R}(\mathbb{D}^{<s>})$?

Theorem 1 [–]

We give the defining equations of $\mathcal{R}(\mathbb{D}^{<s>})$ for the 2 by 2 minors of an m by n matrix of variables for $m \leq n$ or m by m symmetric matrix of variables for $m \leq 5$.

Moreover $\mathcal{R}(\mathbb{D}^{<1>})$ is Cohen-Macaulay for the case of m by n matrix of variables.

With Theorem 1, we recover what the Rees algebra of $\mathbb{D}^{<s>}$ is, i.e. we get information about the blow up, not just the special fiber of the blowup.

1. Defining equations of Rees Algebra

- **For the $\mathbb{D}$ diagonal ideal of S**
- **write $X = [x_{ij}]$, $Y = [y_{ij}]$, $T = [t_{ij}]$**
- **m by n matrices, $m \leq n$**
- **$S = k[X, Y]/(I_{u_1}(X), I_{u_2}(Y))$, $u_1 \geq u_2$ and**

$$S^l \xrightarrow{\phi} S^{mn} \longrightarrow \mathbb{D} \longrightarrow 0.$$

$$0 \to (\mathbf{image}(\phi)) = J \longrightarrow \mathbf{Sym}(S^{mn}) = S[T] \longrightarrow \mathbf{Sym}(\mathbb{D}) \to 0.$$

- J is the ideal generated by the entries of the row vector

$[t_{11}, t_{12}, ..., t_{1n}, ..., t_{mn}] \cdot \phi$.

$$\mathbf{Sym}(\mathbb{D}) \cong S[T]/J.$$

- We can see that J is generated by linear forms in the variables t_{ij}.

- We write $\mathcal{R}(\mathbb{D}) = S[T]/K$, $J \subset K$.

- In general K is not generated by linear forms.

- When $I_{u_i}(X) = I_2(X)$, $\mathcal{F}(\mathbb{D}^{<s>}) = S[T]/I_{s+2}(T)$.

Theorem 2 [–]

$\mathcal{R}(\mathbb{D}^{<s>}) = S[T]/K$ **where** $K = (J, L, I_{s+2}(T))$**, when** X **is a** m **by** n **matrix of variables with** $m \leq n$ **or** m **by** m **symmetric matrix of variables with** $m \leq 5$**. Moreover** $\mathcal{R}(\mathbb{D}^{<1>})$ **is Cohen-Macaulay for the case of** m **by** n **matrix of variables.**

- **Here** L **is an ideal not contained in** J**.**

Proof:

- **Use induction on the size of matrix.**

- **Find a non zero-divisor of** $S[T]/K$ **then localize at it to get to a smaller case.**

- **We compute the Groebner basis of** K **and we get the initial ideals. This way, we find the non zero-divisor.**

Example: **Let** $2 = u_i < m = 3 = n$**.**

- **Write** $\mathcal{R}(\mathbb{D}^{<2>}) = S[T]/K = k[X,Y,T]/\mathcal{K}$. **Then**

$$\begin{aligned}
\mathcal{K} \;=\; & (I_3(X), I_2(Y), \\
& \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ t_{21} & t_{22} & t_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix}, \\
& \begin{vmatrix} t_{ij} & t_{lk} \\ x_{ij} - y_{ij} & x_{lk} - y_{lk} \end{vmatrix}, \\
& \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ y_{21} & y_{22} & y_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix})
\end{aligned}$$

and

$$\begin{aligned}\textbf{in}(\mathcal{K}) \;=\; &(x_{31}x_{22}x_{13},\; y_{21}y_{12},\, y_{21}y_{13},\, y_{22}y_{13},\, y_{31}y_{12},\, y_{31}y_{13},\, y_{31}y_{22},\, y_{31}y_{23},\, y_{32}y_{13},\, y_{32}y_{23},\\
&t_{32}x_{33},\, t_{31}x_{32},\, t_{31}x_{33},\, t_{23}x_{33},\, t_{23}x_{32},\, t_{23}x_{31},\, t_{23}x_{22}x_{13}y_{31},\, t_{23}x_{22}x_{13}y_{11}y_{32},\\
&t_{23}x_{22}x_{13}y_{11}y_{33}, t_{23}x_{22}x_{13}y_{21}y_{32}, t_{23}x_{22}x_{13}y_{21}y_{33},\, t_{22}x_{33},\, t_{22}x_{32},\, t_{22}x_{31},\\
&t_{22}x_{23},\, t_{22}x_{22}x_{13}y_{31},\, t_{22}x_{22}x_{13}y_{11}y_{32},\, t_{22}x_{22}x_{13}y_{11}y_{33},\, t_{22}x_{22}x_{13}y_{21}y_{32},\\
&t_{22}x_{22}x_{13}y_{21}y_{33},\, t_{21}x_{33},\, t_{21}x_{32},\, t_{21}x_{31},\, t_{21}x_{22},\; t_{21}x_{23},\, t_{21}x_{13}x_{21}y_{32},\\
&t_{21}x_{13}x_{21}y_{12}y_{33},\, t_{21}x_{13}x_{21}y_{22}y_{33},\, t_{21}x_{13}y_{32},\, t_{21}x_{13}y_{12}y_{33},\, t_{21}x_{13}y_{22}y_{33},\\
&t_{13}x_{33},\, t_{13}x_{32},\, t_{13}x_{31},\, t_{13}x_{23},\, t_{13}x_{22},\, t_{13}x_{21},\\
&t_{12}x_{33},\, t_{12}x_{32},\, t_{12}x_{31},\, t_{12}x_{21},\; t_{12}x_{22},\, t_{12}x_{23},\, t_{12}x_{13},\\
&t_{11}x_{31},\, t_{11}x_{32},\, t_{11}x_{33},\; t_{11}x_{21},\; t_{11}x_{22},\, t_{11}x_{23},\; t_{11}x_{12},\, t_{11}x_{13},\, t_{11}t_{22}y_{33}).\end{aligned}$$

2. Cohen-Macaulayness of Rees Algebra

Theorem 3 [–]

$\mathcal{R}(\mathbb{D}^{<1>})$ is Cohen-Macaulay for the case of m by n matrix of variables.

- **To show Cohen-Macaulayness, we notice that the initial ideals of the defining ideal of the Rees algebra are square-free monomial ideals. Once we show the initial ideal is Cohen-Macaulay then we have the defining ideal is Cohen-Macaulay.**

Definition: I **is a square-free monomial ideal of** $R = k[x_1, ..., x_n]$ **and** $I = (f_1, ..., f_l)$**, then the Alexander dual ideal** I^* **of** I **is** $\cap_i P_{f_i}$ **, where for any square-free monomial** $f = x_{i_1} \cdots x_{i_r}$**,** $P_f = (x_{i_1}, ..., x_{i_r})$.

Example: Let $I = (ab, bc, cd) \subset k[a, b, c, d]$, then

$I^* = (a, b) \cap (b, c) \cap (c, d) = (ac, bc, bd)$.

Theorem 4 [Eagon-Reiner]

R/I is Cohen-Macaulay $\Leftrightarrow$ I^* has a linear free resolution.

- With Theorem 4, we just need to show that $(\text{in}(\mathcal{L}))^*$, the Alexander dual ideal of in$(\mathcal{L})$, has a linear free resolution. This will give us the Cohen-Macaulayness of $\mathcal{R}(\mathbb{D}^{<s>})$.

Definition: Let

$$\mathbf{F} : ... \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow ... \longrightarrow F_0$$

be a minimal homogeneous free resolution of a graded module M over a ring $R = k[x_1, ..., x_n]$ with $F_i = \oplus_j R(-a_{ij})$.

• **The regularity of M, reg$(M) = \max_i\{a_{ij} - i\}$.**

Notice

If all of the minimal homogeneous generators of I have the same degree d, then I has a linear free resolution $\Leftrightarrow$ reg$(I) = d$.

Lemma 5

The Alexander dual ideal $(\text{in}(\mathcal{L}))^*$ of $(\text{in}(\mathcal{L}))$is generated in degree d and $d = \text{reg}((\text{in}(\mathcal{L}))^*)$, i.e. $(\text{in}(\mathcal{L}))^*$ has a linear free resolution.

• **There exists a filtration of $(\text{in}(\mathcal{L}))^*$ such that the quotients are linear.**

We use this filtration to prove Lemma 5.

3. Summary

Theorem 1 [–]

We give the defining equations of $\mathcal{R}(\mathbb{D}^{<s>})$ for 2 by 2 minors of m by n matrix of variables or m by m symmetric matrix of variables with $m \leq 5$. Moreover $\mathcal{R}(\mathbb{D}^{<1>})$ is Cohen-Macaulay for the case of m by n matrix of variables.

- **The next step is to work on any m and any size minors.**

Conjecture 1

For the case of symmetric matrix, $\mathcal{R}(\mathbb{D}^{<s>})$ has very similar structor as the case of m by n matrix of variables.

Conjecture 2

$\mathcal{R}(\mathbb{D}^{<s>})$ **is Cohen-Macaulay.**

References

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