The Unimodality of Pure O-Sequences of Type Three in Three Variables

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Motivation

- The study of pure O-sequences began in 1977 with the work of Stanley. It has connections to f-vectors of pure multicomplexes, matroid simplicial complexes, and error-correcting codes, among other areas.
- There is a bijective correspondence between pure O-sequences and the Hilbert functions of Artinian level monomial algebras.
- Hibi showed that all pure O-sequences are flawless and the "first half" is non-decreasing. Hausel showed the "first half" is differentiable.

Which sequences are guaranteed to be unimodal?

- Macaulay's Theorem immediately gives that all Artinian algebras in two variables have unimodal Hilbert functions.
- All monomial complete intersections have unimodal Hilbert functions, due to the Weak Lefschetz property (WLP) (Stanley, Watanabe, Reid-Roberts-Roitman).
- All Artinian level monomial algebras of type two in three variables have unimodal Hilbert functions, due to the WLP (BMMNZ).

Which sequences can fail to be unimodal?

- Stanley found $(1, 505, 2065, 3395, 3325, 3493)$ is a non-unimodal pure O-sequence.
- If $r \geq 3$ and M a positive integer, then there exists a pure O-sequence in r variables which has M peaks, and thus is not unimodal (BMMNZ).
- The smallest known type for an Artinian level monomial algebra that fails to have a unimodal Hilbert function is 14 (BMMNZ).
- The WLP can fail for Artinian level monomial algebras in three variables when the type ≥ 3, regardless of the characteristic (Brenner-Kaid).

We will give a positive answer for the unimodality of the Hilbert functions in the smallest open case, that of Artinian level monomial algebras of type three in three variables.

Definitions

- An *order ideal* is a non-empty set X of (monic) monomials such that if $M \in X$ and N is a monomial dividing M, then $N \in X$.
- The *h-vector* of X is the sequence $\underline{h}= (h_0 = 1, h_1, \cdots, h_e)$ which counts the monomials of X in each degree.
- An order ideal is *pure* if, after ordering by divisibility, all maximal monomials of X have the same degree. The *type* of an O-sequence is the number of maximal monomials in the order ideal.
- A *pure* O-sequence is the h-vector of a pure order ideal.

Classification Theorem

Theorem Let R = k[x,y,z] and let I be a monomial ideal such that R/I is level of type 3. Then I has one of the following four forms, up to a change of variables. (Without loss of generality, we will assume that $a \geq \alpha_2 \geq \alpha_1$, $b \geq \beta_2 \geq \beta_1$, $c \geq \gamma_2 \geq \gamma_1$.)

1. $(x^a, x^{\alpha_2}z^{\gamma_1}, x^{\alpha_1}z^{\gamma_2}, z^c, y^{\beta_1}z^{\gamma_2}, y^{\beta_2}z^{\gamma_1}, y^b)$ where $a + b + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \alpha_1 + \beta_1 + c$.
   The Hilbert function of R/I is
   $$H(a, b, \gamma_1) + H(\alpha_2, \beta_2, \gamma_2 - \gamma_1)(-\gamma_1) + H(\alpha_1, \beta_1, c - \gamma_2)(-\gamma_2).$$
2. $(x^a, x^{\alpha_2}z^{\gamma_1}, x^{\alpha_1}z^{\gamma_2}, z^c, y^{\beta_2}z^{\gamma_1}, y^b, x^{\alpha_1}y^{\beta_1}z^{\gamma_1})$ where $a + b + \gamma_1 = \alpha_2 + \beta_1 + \gamma_2 = \alpha_1 + \beta_2 + c$.
   The Hilbert function of R/I is
   $$H(a, b, \gamma_1) + H(\alpha_1, \beta_2, c - \gamma_1)(-\gamma_1) + H(\alpha_2 - \alpha_1, \beta_1, \gamma_2 - \gamma_1)(-\alpha_1 - \gamma_1).$$
3. $(x^a, x^{\alpha_2}z^{\gamma_1}, x^{\alpha_2}y^{\beta_1}, z^c, y^{\beta_2}z^{\gamma_2}, x^{\alpha_1}y^{\beta_2}, y^b)$ where $a + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + c = \alpha_1 + b + \gamma_2$.
   The Hilbert function of R/I is
   $$H(\alpha_2, \beta_2, c) + H(a - \alpha_2, \beta_1, \gamma_1)(-\alpha_2) + H(\alpha_1, b - \beta_2, \gamma_2)(-\beta_2).$$
4. $(x^a, x^{\alpha_2}z^{\gamma_1}, z^c, y^{\beta_1}z^{\gamma_2}, y^b, x^{\alpha_1}y^{\beta_2}, x^{\alpha_1}y^{\beta_1}z^{\gamma_1})$ where $a + \beta_2 + \gamma_1 = \alpha_1 + b + \gamma_2 = \alpha_2 + \beta_1 + c$.
   The Hilbert function of R/I is
   $$H(\alpha_1, b, \gamma_1)+H(a - \alpha_1, \beta_2, \gamma_1)(-\alpha_1) + H(\alpha_2, \beta_1, c - \gamma_1)(-\gamma_1)+ H(\alpha_1, b - \beta_1, \gamma_2 - \gamma_1)(-\beta_1 - \gamma_1).$$

Proof sketch: Using Macaulay's theory of inverse systems and examining all possible inequalities between the exponents, we can obtain this result.

Licci Theorem

Theorem We have that:

1. The ideals of the form (1) and (2) in the classification theorem are licci
2. The ideals of the form (3) and (4) in the classification theorem are not licci.

We conjecture that ideals (3) and (4) are glicci.

Proof sketch: For 1, we notice that ideals (1) and (2) can be decomposed into the form $I = z^{\gamma_1} \cdot L + (x^a, y^b)$, with $L = f \cdot T + J$, where T and J are complete intersections. Lemma 2.5 of Huneke and Ulrich gives us the construction of the CI-links which link L to T. Similarly, Lemma 2.5 gives the construction of the CI-links which link I to L, thus we have that I is CI-linked to a complete intersection (namely T).
For 2, we have that ideals (3) and (4) decompose as
$$I = (x^a, y^b, z^c) + (x^{\alpha_2}z^{\gamma_1}, x^{\alpha_2}y^{\beta_1}, y^{\beta_2}z^{\gamma_2}, x^{\alpha_1}y^{\beta_2}) \text{ and}$$
$$I = (x^a, y^b, z^c) + (x^{\alpha_2}z^{\gamma_1}, y^{\beta_1}z^{\gamma_2}, x^{\alpha_1}y^{\beta_2}, x^{\alpha_1}y^{\beta_1}z^{\gamma_1}) \text{ respectively.}$$
Since the second piece of each ideal has height at least two, Theorem 2.4 of Huneke and Ulrich tells us that they are not licci.

Main Theorem

Theorem Let R = k[x,y,z] and let I be a monomial Artinian ideal such that R/I is level of type three. Then the Hilbert function of R/I is unimodal.

Proof sketch: We will only sketch the proof for ideals (1) and (2) here. Ideals (3) and (4) break down slightly differently, but the basic outlines of those proofs are very similar.

<u>Lemma</u> Let $\mathfrak{a} = (x^a, y^b, z^c)$ be a complete intersection, and $H(a, b, c)$ be its Hilbert function. Let ΔH be the first difference of this Hilbert function. Then
$$\Delta H = H(a, c) - H(a, c)(-b).$$
Any permutation of a, b, c is equally valid.

This lemma breaks down the Hilbert functions to the form
$$\begin{aligned}\Delta H &= [H(\rho, \tau) + H(\eta, \kappa)(-\tau) + H(\nu, \xi)(-\kappa - \tau)] \\ &\quad -[H(\rho, \tau)(-\sigma) + H(\eta, \kappa)(-\mu - \tau) + H(\nu, \xi)(-\omega - \kappa - \tau)] \\ &:= P - N.\end{aligned}$$

<u>Claim 1:</u> If $P(t_1) < N(t_1)$, then $\Delta P(t) \leq 0$ for $t \geq t_1$.
Proof sketch: Since the lemma reduces the Hilbert functions to Hilbert functions of complete intersections in two variables, we know what each piece of P and N looks like. Thus, we are left to examine the shifts and respective values of $\Delta H(t)$ for $t \geq t_1$ to show that $\Delta P(t) \leq 0$.

<u>Claim 2:</u> There do not exist degrees $t_1 < t_2$ such that $P(t_1) < N(t_1)$ and $N(t_2) < P(t_2)$.
Proof sketch: Since we know that P does not increase after t_1, it only remains to check when N decreases faster than P, and we are left to show that P cannot become greater than N in these cases.

Sources

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