# Enumerations deciding the weak Lefschetz property

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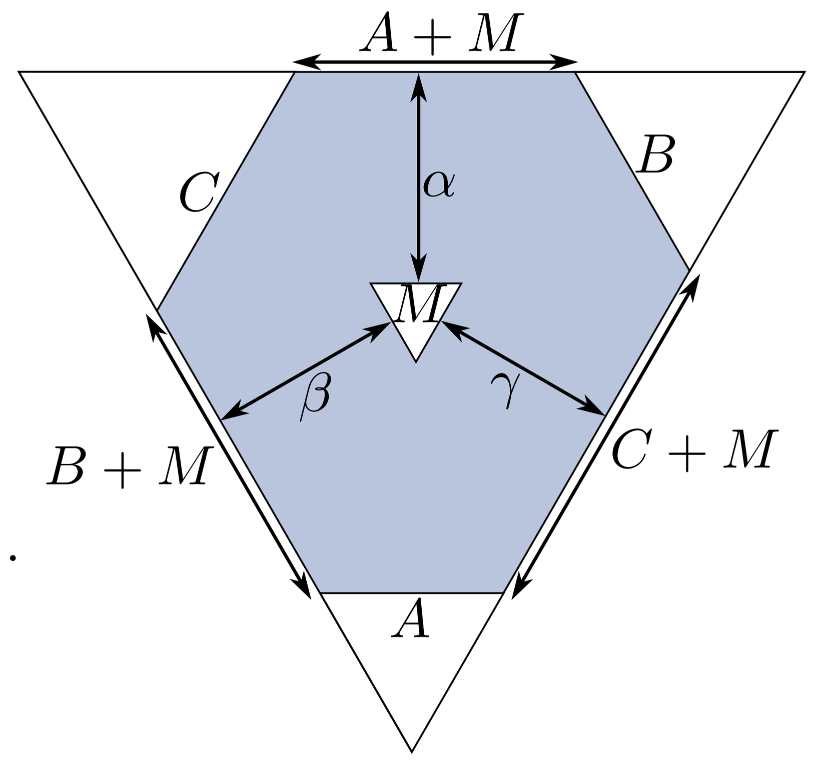
An artinian $K$-algebra $A$ is said to have the **weak Lefschetz property** if there exists a linear form $\ell$ such that, for all integers $d$, the map

$$\times\ell : [A]_d \to [A]_{d+1}$$

has maximal rank. In this case, a form $\ell$ is called a **Lefschetz element** of $A$.

Consider $I=(x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$ in $R=K[x,y,z]$. We associate to $I$ the following **punctured hexagon**:




where

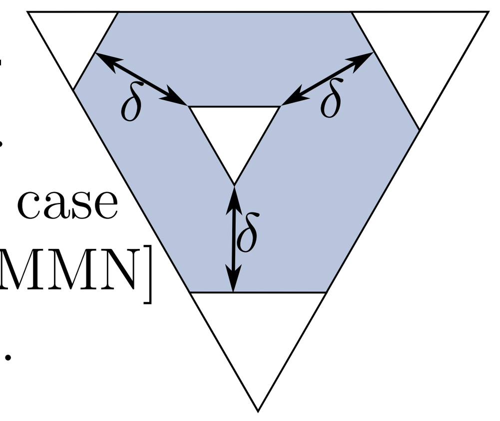
$$s := \tfrac{1}{3}(a+b+c+\alpha+\beta+\gamma)-2,$$
$$A := s+2-a,$$
$$B := s+2-b,$$
$$C := s+2-c, \text{ and}$$
$$M := s+2-(\alpha+\beta+\gamma).$$

By Lindström-Gessel-Viennot Theorem the enumeration of signed lozenge tilings of this region is given by

$$D := \det_{1\le i,j\le C+M} \begin{cases} \binom{c}{A+j-i} & \text{if } 1\le i\le C,\\ \binom{\gamma}{A+C-\beta+j-i} & \text{if } C+1\le i\le C+M.\end{cases}$$

**Theorem:** $R/I$ has the weak Lefschetz property iff $D \not\equiv 0$ mod char $K$.

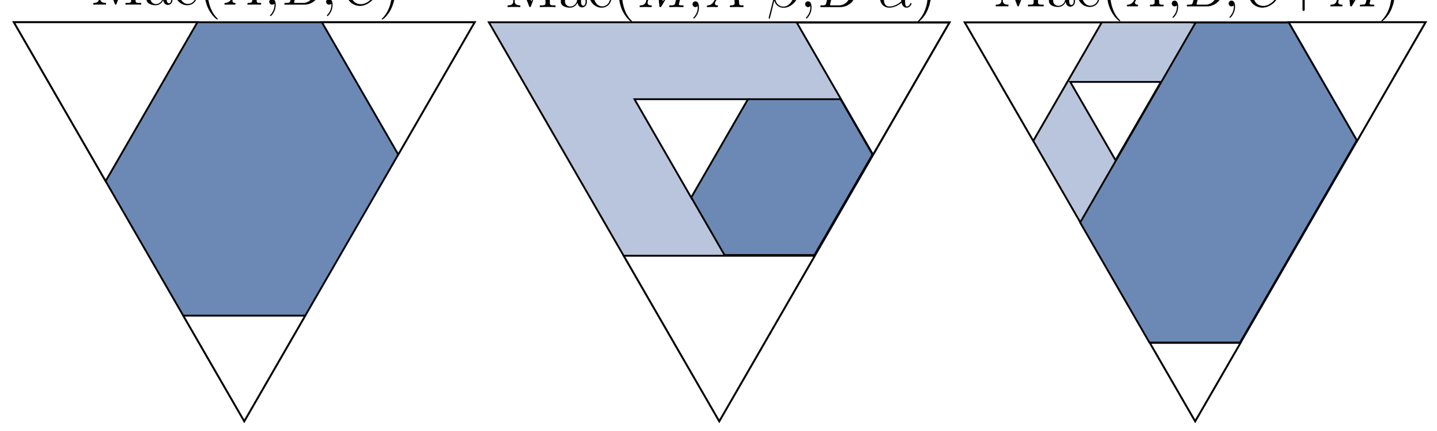
When the puncture is **gravity-central**, the ideal is **level**. This corresponds to a case studied in both [MMN] and [CN-09].




Several cases correspond to tilings of *unpunctured* hexagons and so are enumerated by **MacMahon's formula**:

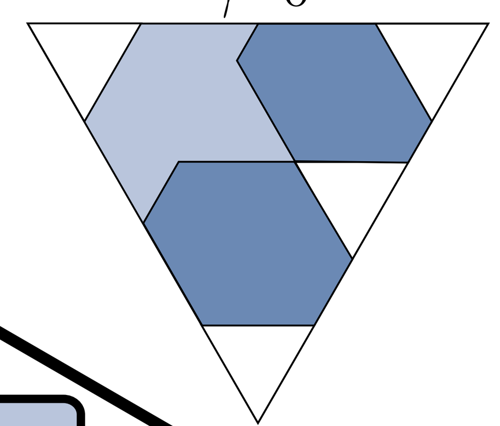
$$\mathrm{Mac}(a,b,c) := \frac{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(a+b+c)}{\mathcal{H}(a+b)\mathcal{H}(a+c)\mathcal{H}(b+c)},$$

where $\mathcal{H}(n) = \prod_{i=0}^{n-1} i!$ is the **hyperfactorial** of $n$.




The case $M$=0 degenerates to the complete intersection case studied in [CN-09], [LZ], and [CGJL].

When $\gamma$=0, then the determinant can be computed explicitly using a product of Mahonian determinants and hyperfactorials.



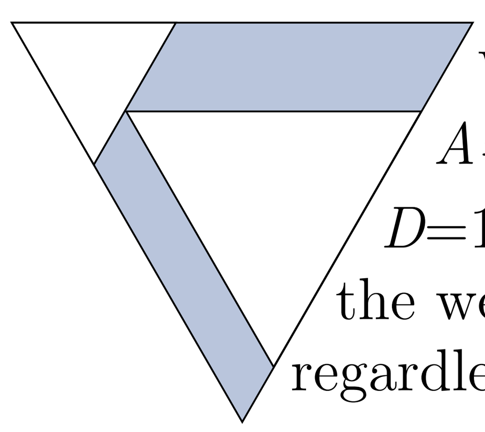

When the puncture has even side-length, i.e., $M$ is even, then all lozenge tilings of the punctured hexagon have the same sign. This establishes the weak Lefschetz property, in characteristic 0, for many new cases.

**Theorem:** $R/I$ has the weak Lefschetz property in characteristic 0, if $M$ is even.

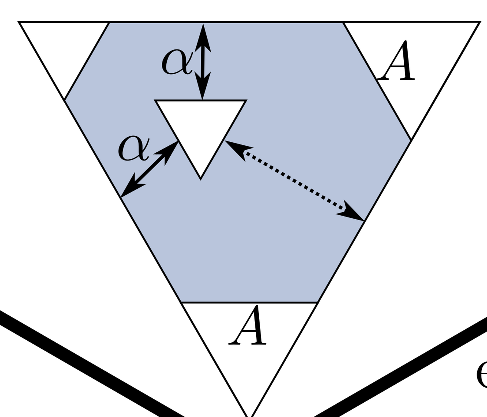
Consider a punctured hexagon symmetric about a median of the outer triangle, i.e., $A$=$B$ and $\alpha$=$\beta$.

**Theorem:** Symmetric punctured hexagons with $C$ and $M$ odd have $D$=0; that is, the associated algebra $R/I$ *never* has the weak Lefschetz property.

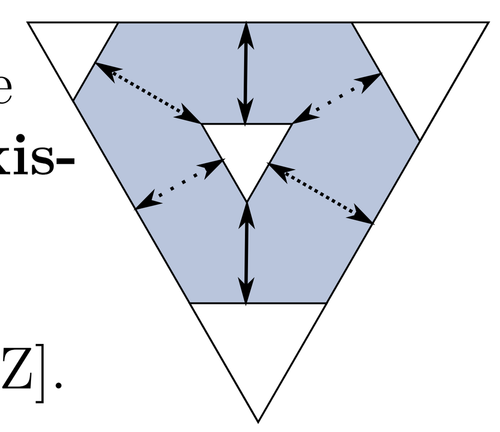
Moreover, we have a conjecture of a closed formula for $D$, when $C$ or $M$ is even; it is a polynomial in $M$ with only linear factors.

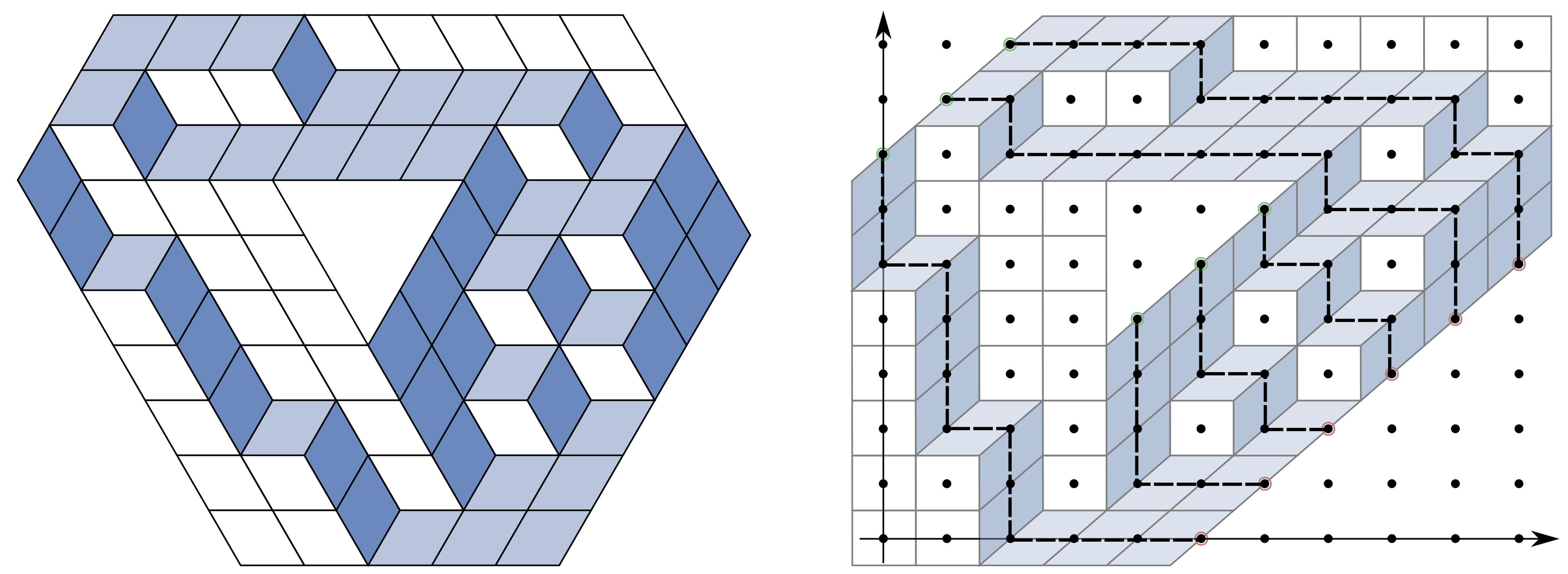


When $A$=$B$=$\gamma$=0, then $D$=1 and $R/I$ *always* has the weak Lefschetz property, regardless of the characteristic of $K$.




When the puncture is **axis-central**, then the desired enumeration is explicitly computed in [CEKZ].





Lozenge tilings of punctured hexagons are in bijection with families of non-intersecting lattice paths on a particular lattice (the above pair are associated). The **sign of a lozenge tiling** is given by the signature of the permutation of the endpoints of the associated family of non-intersecting lattice paths.

In the case when the ideal is associated to a punctured hexagon, then deciding the presence of the weak Lefschetz property is equivalent to other problems in algebra, combinatorics, and algebraic geometry (we use results from [BK] and an undescribed connection given in [CN-11]).

**Theorem:** Suppose the ideal $I$ can be associated to a punctured hexagon, then the following are equivalent:

(i) The algebra $R/I$ has the weak Lefschetz property;
(ii) the Castelnuovo-Mumford regularity of $R/(I, x+y+z)$ is $s$;
(iii) the enumeration ($D$) of signed lozenge tilings of the punctured hexagon modulo the characteristic of $K$ is non-zero; and
(iv) the enumeration of signed perfect matchings of the bipartite graph associated to the punctured hexagon modulo the characteristic of $K$ is non-zero.

If the characteristic of $K$ is zero, then there is one further equivalent condition:

(v) The generic splitting type of syz $I$ is ($s$+2, $s$+2, $s$+2).

Note that the monomial almost complete intersections *not* associated to punctured hexagons can be shown, using other arguments, to always have the weak Lefschetz property in characteristic 0.

## References: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