Local Volumes of Divisors over Normal Algebraic Varieties

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The Goal

Study asymptotic invariants of complex normal isolated singularities measuring the growth of classical plurigenera-type invariants of the singularities.

The Set-up

Fix (X,x) a normal isolated singularity of dimension n and fix $\pi : (\widetilde{X}, E) \to (X,x)$ a log-resolution, E being reduced. The $m-$th Morales plurigenus of (X,x) is the local invariant

$$\lambda_m(X,x) = \dim \frac{\mathcal{O}_X(mK_X)}{\pi_*\mathcal{O}_{\widetilde{X}}(m(K_{\widetilde{X}}+E))}.$$

- (X,x) smooth $\Rightarrow \lambda_m(X,x) = 0$.
- $\lambda_m(X,x)$ is independent of π.
- Asymptotically, $\lambda_m(X,x) = O(m^n)$.

Define the volume of (X,x) as

$$\boxed{\mathrm{vol}(X,x) =_{\mathrm{def}} \limsup_{m\to\infty} \frac{\lambda_m(X,x)}{m^n/n!} < \infty.}$$

The history of the problem

Wahl considered $\mathrm{vol}(X,x)$ for isolated surface singularities. He showed

- $\mathrm{vol}(X,x) = 0 \Leftrightarrow (X,x)$ is log-canonical.
- $\mathrm{vol}(X,x)$ is a characteristic number of the link of the singularity, computable from the dual graph of a good resolution.
- $f : (X,x) \to (Y,y)$ finite and $f^{-1}\{y\} = \{x\}$, then $\mathrm{vol}(X,x) \geq (\deg f)\cdot \mathrm{vol}(Y,y)$.
  1. We have equality when f is unramified over $Y \setminus \{y\}$.
  2. (X,x) log-canonical $\Rightarrow (Y,y)$ log-canonical.
  3. $(X,x) = (Y,y)$ and $\deg f \geq 2$ imply (X,x) log-canonical.

If P is the $\pi-$nef part of the relative Zariski decomposition of $K_{\widetilde{X}} + E$, Wahl shows that $\mathrm{vol}(X,x) = -P\cdot P$. With this idea in mind, drawing on the theory of $b-$divisors, in recent work, Boucksom, de Fernex and Favre define a volume $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$ for normal isolated singularities in arbitrary dimension. They show $\mathrm{vol}_{\mathrm{BdFF}}(X,x) = \mathrm{vol}(X,x)$ in the $\mathbb{Q}$-Gorenstein case, but the two volumes may differ in general.

Local volumes

A local cohomology perspective on $\lambda_m(X,x)$ shows that $\mathrm{vol}(X,x)$ is a particular case of a more general problem. Let $\pi : X' \to X$ be proper birational, X is normal, $n-$dimensional and $x \in X$. Denote by $i : X \setminus \{x\} \to X$ the natural embedding. For D Cartier on X', let

$$h^1_x(D) = \dim H^1_{\{x\}}(X, \pi_*\mathcal{O}_{X'}(D)) = \dim \frac{i_*i^*\pi_*\mathcal{O}(D)}{\pi_*\mathcal{O}(D)}.$$

Define the local volume of D over x as

$$\boxed{\mathrm{vol}_x(D) = \limsup_{m\to\infty} \frac{h^1_x(mD)}{m^n/n!}.}$$

For isolated singularities,

$$\lambda_m(X,x) = h^1_x(m(K_{\widetilde{X}}+E))$$
$$\mathrm{vol}(X,x) = \mathrm{vol}_x(K_{\widetilde{X}}+E).$$

Theorem (properties of vol_x)

- $\mathrm{vol}_x(D)$ is finite.
- $\mathrm{vol}_x(D)$ is local around x, i.e., invariant upon base change to any open neighborhood of x.
- $f : X'' \to X'$ proper birational $\Rightarrow$ $\mathrm{vol}_x(D) = \mathrm{vol}_x(f^*D)$.
- $\mathrm{vol}_x(aD) = a^n\cdot \mathrm{vol}_x(D)$ for $a \in \mathbb{N}$.
- $\mathrm{vol}_x(D+P) = \mathrm{vol}_x(D)$ if $P\cdot C = 0$ for any curve C contracted by π. Thus vol_x descends to $N^1(X'/X)_{\mathbb{Q}}$.
- vol_x is locally uniformly continuous on $N^1(X'/X)_{\mathbb{Q}}$, so it extends continuously over $\mathbb{R}$.
- X' normal, $\mathrm{vol}_x(D) = 0 \Rightarrow h^1_x(mD) = 0\ \forall m \geq 0$.
- $\lim$ can replace $\limsup$ in the definition of $\mathrm{vol}_x(D)$.
- vol_x is log-convex when working with divisors supported in $\pi^{-1}(x)$ and not otherwise.
- $f : X \to Y$ finite morphism of normal varieties, $y \in Y$, $f^{-1}\{y\} = \{x_1,\ldots,x_k\}$, $\rho : Y' \to Y$ and $\pi : X' \to X$ proper birational, $f' : X' \to Y'$ a lift of f and D Cartier on Y', then $(\deg f)\cdot \mathrm{vol}_y(D) = \sum_{i=1}^k \mathrm{vol}_{x_i}(f'^*D)$.
- vol_x can be realized as a normalized volume of a (finite) difference of two infinite nested polytopes.

Isolated singularities

Let (X,x) again denote a normal complex isolated singularity of dimension n and $\pi : (\widetilde{X}, E) \to (X,x)$ a log-resolution with E the reduced fiber over x. The properties of vol_x help extend many of the results proved by Wahl for surfaces and allow us to compare $\mathrm{vol}(X,x)$ and $\mathrm{vol}_{\mathrm{BdFF}}(X,x)$.

- **Vanishing:** If (X,x) is $\mathbb{Q}-$Gorenstein, TFAE
  1. $\mathrm{vol}(X,x) = 0$.
  2. $\lambda_m(X,x) = 0$ for all $m \in \mathbb{N}$.
  3. $\mathcal{O}_X(mK_X) = \pi_*\mathcal{O}_{\widetilde{X}}(m(K_{\widetilde{X}}+E))$ for all $m \in \mathbb{N}$.
  4. (X,x) is log-canonical.

  Counterexamples exist in the non $\mathbb{Q}-$Gorenstein case.
- **Finite maps:** $f : (X,x) \to (Y,y)$ finite and $f^{-1}\{y\} = \{x\}$, then
  $$\mathrm{vol}(X,x) \geq (\deg f)\cdot \mathrm{vol}(Y,y).$$
  1. We have equality when f is unramified over $Y \setminus \{y\}$.
  2. $\mathrm{vol}(X,x) = 0 \Rightarrow \mathrm{vol}(Y,y) = 0$.
  3. $(X,x) = (Y,y)$ and $\deg f \geq 2$ imply $\mathrm{vol}(X,x) = 0$.
- **Examples:**
  1. The above results can be used to show that $\mathrm{vol}(X,x) = 0$ for quotient singularities, for toric or cusp isolated singularities and of course for log-canonical $\mathbb{Q}-$Gorenstein singularities.
  2. **Cone singularities.** Let (X,x) be the cone over the polarized smooth projective variety (V,H) of dimension $n-1$. Then
  $$\boxed{\mathrm{vol}(X,x) = n\cdot \int_0^\infty \mathrm{vol}(V, K_V - t\cdot H)dt.}$$
  The volume under the integral is the volume of Cartier divisors on the projective variety V.
  3. **Quasi-homogeneous hyper-surface singularities.** Let (X,x) be the isolated singularity associated to a quasi-homogeneous polynomial $f \in \mathbb{C}[X_0,\ldots,X_n]$ of type $(r_0,\ldots,r_n)$. Watanabe computes:
  $$\mathrm{vol}(X,x) = \begin{cases} 0 & \text{, if } r_0+\ldots+r_n > 1 \\ \frac{(1-r_0-\ldots-r_n)^n}{r_0\cdot\ldots\cdot r_n} & \text{, otherwise} \end{cases}.$$
- **Comparison:** We have
  $$\mathrm{vol}_{\mathrm{BdFF}}(X,x) \geq \mathrm{vol}(X,x).$$
  The equality holds in the $\mathbb{Q}-$Gorenstein case, but it fails for some cone singularities.

Pathologies and questions

- **Topological behavior.** On surfaces, Wahl shows that $\mathrm{vol}(X,x)$ is a topological invariant of the link of the singularity. This is no longer the case in arbitrary dimension, but none of our examples are $\mathbb{Q}-$Gorenstein.
- **Irrational examples.** We can construct cone singularities (X,x) such that $\mathrm{vol}_{\mathrm{BdFF}}(X,x) > \mathrm{vol}(X,x)$ with both irrational. We do not know of a $\mathbb{Q}-$Gorenstein irrational example.
- **Boundedness.** It is known that the set of volumes of smooth projective varieties of general type of dimension n is bounded from below. This is not the case in general for volumes of isolated singularities, but the question is known to be true in the Gorenstein case in dimension two and open in the Gorenstein case in arbitrary dimension.

Selected references

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