Chern Number Formulas and Consequences

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Abstract

We establish a formula for the Chern number in a d-dimensional local ring involving modified Koszul complex and specifically give the formula in dimension one and two. As applications we unify several results for Chern number in local rings of dimension ≤ 2.

Introduction

Let $(R, \mathfrak{m})$ be a local ring and I be an $\mathfrak{m}$-primary ideal. Then a sequence of ideals $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ is called an I-admissible filtration if there exists $k \in \mathbb{N}$ such that for all $n, m \in \mathbb{Z}$ the following three conditions hold

(a) $I_{n+1} \subseteq I_n$,
(b) $I_m I_n \subseteq I_{n+m}$,
(c) $I^n \subseteq I_n \subseteq I^{n-k}$ for all $n \geq k$.

Marley showed that if $\mathcal{I}$ is an I-admissible filtration then the function $H_{\mathcal{I}}(n) = \lambda(R/I_n)$ where λ denotes length as R-module coincides with a polynomial $P_{\mathcal{I}}(x) \in \mathbb{Q}[x]$ of degree d for large n. This polynomial is written as

$$P_{\mathcal{I}}(x) = e_0(\mathcal{I})\binom{x+d-1}{d} - e_1(\mathcal{I})\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{I})$$

and it is called the Hilbert polynomial associated with the I-admissible filtration $\mathcal{I}$. The integers $e_i(\mathcal{I})$ for $i = 1, \ldots, d$ are called the Hilbert coefficients of $\mathcal{I}$. The coefficient $e_1(\mathcal{I})$ is called the Chern number of $\mathcal{I}$.

Formula for the Chern number involving modified Koszul complex

Let $(R, \mathfrak{m})$ be a d-dimensional local ring and $\{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration. Let $x_1, \ldots, x_d \in I$. Define for $n \in \mathbb{Z}$, the modified Koszul complex $C_.(n, \mathcal{I})$

$$0 \longrightarrow R/I_{n-d} \longrightarrow (R/I_{n-d+1})^{\binom{d}{1}} \longrightarrow \cdots \longrightarrow (R/I_{n-1})^{\binom{d}{d-1}} \longrightarrow (R/I_n) \longrightarrow 0 \qquad (1)$$

with $I_n = R$ for $n \leq 0$ and the differentials are induced by the differentials of the Koszul complex $K_. = K_.(x_1, \ldots, x_d; R)$. In other words we have an exact sequence of complexes

$$0 \longrightarrow K^{(n)}(\mathcal{I}) \longrightarrow K_.(x_1, \ldots, x_d) \longrightarrow C_.(n, \mathcal{I}) \longrightarrow 0 \qquad (2)$$

where $K^{(n)}(\mathcal{I})$ is a subcomplex of the Koszul complex given by

$$K^{(n)}(\mathcal{I}): \quad 0 \longrightarrow I_{n-d} \longrightarrow (I_{n-d+1})^{\binom{d}{1}} \longrightarrow \cdots \longrightarrow (I_{n-1})^{\binom{d}{d-1}} \longrightarrow I_n \longrightarrow 0.$$

The Euler characteristic of $C_.(n, \mathcal{I})$ is defined as

$$\chi(C_.(n, \mathcal{I})) = \sum_{i=0}^{d} (-1)^i \lambda(H_i(C_.(n, \mathcal{I}))).$$

Then from the exact sequence (2) we have

$$\chi(K_.) = \chi(K^{(n)}(\mathcal{I})) + \chi(C_.(n, \mathcal{I})).$$

Main Theorem. Let $(R, \mathfrak{m})$ be a d-dimensional local ring. Let I be an $\mathfrak{m}$-primary ideal and $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration. Then

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \chi(K^{(n)}(\mathcal{I})).$$

Sketch of Proof. By Using the theory of numerical functions we observe that

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \triangle^d [P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)].$$

By Serre's Theorem we have $e_0(\mathcal{I}) = \chi(K_.)$. Hence from the exact sequence (2) we have

$$\triangle^d [P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \chi(K^{(n)}(\mathcal{I})).$$

References

C.Huneke, *Hilbert functions and symbolic powers,* Michigan Math. J. **34**(2) (1987), 293–318

J. Lipman, *Stable ideals and Arf rings* , Amer. J. Math. **93** (1971), 649 –685.

D. Rees, *$\mathcal{A}$-transforms of local rings and a theorem on multiplicities of ideals* Proc. Cambridge Philos. Soc. **57** (1961), 8–17.

J. D. Sally, *Hilbert coefficients and reduction number* 2, J. Algebraic Geom **1** (1992), 325-333.

Consequences of Main Theorem

Corollary 1. Let $(R, \mathfrak{m})$ be a 1-dimensional local ring and I be an $\mathfrak{m}$-primary ideal and $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration. Let $J = (x) \subseteq I_1$ be a minimal reduction. Then

(a) $e_1(\mathcal{I}) = \sum_{n=1}^{\infty} [\lambda(I_n/JI_{n-1}) - \lambda((0:x) \cap I_{n-1})]$.

(b) If R is Cohen-Macaulay then

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \lambda(I_n/JI_{n-1}).$$

Corollary 2. Let $(R, \mathfrak{m})$ be a 2-dimensional local ring. Let $J = (x, y)$ be a parameter ideal. Let $\mathcal{J} = \{J_n\}$ be a J-admissible filtration and depth $G(\mathcal{J}) \geq 1$ and $x^* \in J_1/J_2$ be a nonzerodivisor in $G(\mathcal{J})$. Then

$$e_1(\mathcal{J}) = e_0(\mathcal{J}) - \lambda(R/J_1) + \sum_{n\geq 2} \left[\lambda\left(\frac{J_n}{JJ_{n-1}}\right) - \lambda\left(\frac{(x:y) \cap J_{n-1}}{(x)J_{n-2}}\right)\right]$$

Corollary 3. Let $(R, \mathfrak{m})$ be a 2-dimensional analytically unramified local ring and $J = (x, y)$ be a parameter ideal. Then

$$\overline{e}_1(J) = e_0(J) - \lambda(R/\overline{J}) + \sum_{n\geq 2} \left[\lambda\left(\frac{\overline{J^n}}{J\overline{J^{n-1}}}\right) - \lambda\left(\frac{(x:y) \cap \overline{J^{n-1}}}{(x)\overline{J^{n-2}}}\right)\right].$$

Applications

Rees, 1961

Let $(R, \mathfrak{m})$ be a 1-dimensional local ring. Let I be an $\mathfrak{m}$-primary ideal and $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration. Let $J \subseteq I_1$ be a parameter ideal such that $e_0(J) = e_0(\mathcal{I})$. Then J is a reduction of $\mathcal{I}$.

Lipman, 1971

Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay ring and I be an $\mathfrak{m}$-primary ideal and $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ be an I- admissible filtration. Let $J = (x)$ is a minimal reduction of $\mathcal{I}$. Then for all $n \geq 1$

$$\lambda(I_{n-1}/I_n) \leq e_0(\mathcal{I})$$

and

$$\lambda(I_{n-1}/I_n) = e_0(\mathcal{I}) \text{ if and only if } I_n = xI_{n-1}.$$

Huneke,1987

Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay ring and I be an $\mathfrak{m}$-primary ideal and $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration. Assume that

$$e_1(\mathcal{I}) = e_0(\mathcal{I}) - \lambda(R/I_1).$$

Then

- $H_{\mathcal{I}}(n) = P_{\mathcal{I}}(n)$ for all $n \geq 1$.
- If (x) is a minimal reduction of $\mathcal{I}$ then $I_n = xI_{n-1}$ for all $n \geq 2$.

Huneke, 1987

Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal and $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration. Let $J = (x, y)$ be a minimal reduction of $\mathcal{I}$. Then for all $n \geq 2$

- $\triangle^2[P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \lambda\left(\frac{I_n}{JI_{n-1}}\right) - \lambda\left(\frac{I_{n-1}:J}{I_{n-2}}\right)$.
- $e_1(\mathcal{I}) = e_0(\mathcal{I}) - \lambda(R/I_1) + \sum_{n\geq 2}\left[\lambda\left(\frac{I_n}{JI_{n-1}}\right) - \lambda\left(\frac{I_{n-1}:J}{I_{n-2}}\right)\right]$.

Sally, 1992

Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay ring and I be an $\mathfrak{m}$-primary ideal. Assume that

$$e_1(I) = e_0(I) - \lambda(R/I) + 1.$$

Then

- $H_I(n) = P_I(n)$ for all $n > 1$.
- If (x) is a minimal reduction of I then $\lambda(I^2/xI) = 1$ and $I^3 = xI^2$.