Minimal Free Resolutions of General Points on Cubic Surfaces

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Notation and Definitions

- k an algebraically closed field
- $R = k[x_0, \dots, x_n]$
- $X \subset \mathbb{P}^n_k = \mathbb{P}^n$, I_X its saturated ideal
- The *(graded) Betti numbers* of X are $\beta_{i,j}$ where MFR is

$$0 \to \bigoplus_{j=1}^{m_c} R(-c-j)^{\beta_{c,j}} \to \cdots \to \bigoplus_{j=1}^{m_2} R(-2-j)^{\beta_{2,j}} \to \bigoplus_{j=1}^{m_1} R(-1-j)^{\beta_{1,j}} \to R \to R/I_X \to 0$$

- The *Betti diagram* of X is

	0	1	2	$\cdots$	c
0	1	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\cdots$	$\beta_{c,1}$
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\cdots$	$\beta_{c,2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r-1$	–	$\beta_{1,r-1}$	$\beta_{2,r-1}$	$\cdots$	$\beta_{c,r-1}$
r	–	–	–	$\cdots$	–

- We call r the *regularity of* X, $\operatorname{reg} X$.

Minimal Resolution Conjecture

If $Z \subset \mathbb{P}^n$ is a set of t points in general position then:
- it has Hilbert function $Hf_Z(j) = \min\{\binom{n+j}{j}, t\}$ and
- the MFR of Z has the form

$$0 \to \begin{matrix} R(-(d+n-2))^{\beta_{n-1,d}} \\ \oplus \\ R(-(d+n-1))^{\beta_{n-1,d+1}} \end{matrix} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta 2} \begin{matrix} R(-d)^{\beta_{1,d}} \\ \oplus \\ R(-(d+1))^{\beta_{1,d+1}} \end{matrix} \xrightarrow{\delta_1} R \to R/I_Z \to 0$$

$d = \min\{z \in \mathbb{Z} \mid \binom{z+n}{n} > t\}$

Minimal Resolution Conjecture (Lorenzini)

$\beta_{i,j}\beta_{i-1,j+1} = 0$ for any i, j.

- True in $\mathbb{P}^2$, $\mathbb{P}^3$, $\mathbb{P}^4$ (Gaeta, Geramita-Lorenzini; Ballico-Geramita; Walter) and if $t \gg n$ in $\mathbb{P}^n$ (Hirschowitz-Simpson).
- False in $\mathbb{P}^n$, $n \geq 6$, except possibly $n = 9$ (Eisenbud-Popescu-Schreyer-Walter).

Generalized Minimal Resolution Conjecture

For $Z \subset X$ be t general points on X with $P_X(a-1) \leq t < P_X(a)$, $a \geq \operatorname{reg} X + 1$, the Betti diagram of Z looks like (Mustaţă):

	0 1 etc.
0	Betti diagram
$\vdots$	for X
$\operatorname{reg} X - 1$	
$\vdots$	$\cdots$ 0 $\cdots$
$a-1$	two more
a	nonzero rows

Conjecture (Mustaţă)

$\beta_{i+1,a-1}(Z)\beta_{i,a}(Z) = 0$ for all i.

- Farkas-Mustaţă-Popa have some results for X a curve.
- Holds for general sets of points on a quadric surface in $\mathbb{P}^3$ (Giuffrida-Maggioni-Ragusa).

On Smooth Cubic Surfaces

Let $X \subset \mathbb{P}^n$ be a smooth cubic surface and $Z \subset X$ a set of t general points on X with $P_X(a-1) \leq t < P_X(a)$, where $a \geq \operatorname{reg} X + 1$.
- Mustaţă showed that the MRC holds if $t = P_X(a-1)$ or $t = P_X(a) - 1$.
- Casanellas showed that the MRC holds if $t = m(a), n(a), n(a)+1$, or $m(a)+1$ where
  - $m(a) = \frac{3}{2}a(a-1)$
  - $n(a) = \frac{3}{2}a(a-1) + 2a$

Casanellas used Gorenstein liaison and the mapping cone to prove her result.

Theorem (Migliore-P)

Let $X \subset \mathbb{P}^3$ a cubic surface, smooth or with finitely many double points, and Z_t be a set of $t \geq 7$ general points on X, with $a \in \mathbb{Z}$ defined by $P_X(a-1) < t \leq P_X(a)$. Let $b = t - P_X(a-1)$. Then the MFR for Z_t is

$$0 \longrightarrow \begin{matrix} R(-a-2)^{3(a-b-1)} \\ \oplus \\ R(-a-3)^b \end{matrix} \longrightarrow R(-a-1)^{3(2a-b-1)} \longrightarrow \begin{matrix} R(-3) \\ \oplus \\ R(-a)^{3a-b} \end{matrix} \longrightarrow R \longrightarrow R/I_{Z_t} \longrightarrow 0 \qquad \text{if } P_X(a-1) < t \leq m(a),$$

$$0 \longrightarrow R(-a-3)^b \longrightarrow \begin{matrix} R(-a-1)^{3(2a-b-1)} \\ \oplus \\ R(-a-2)^{3(b-a+1)} \end{matrix} \longrightarrow \begin{matrix} R(-3) \\ \oplus \\ R(-a)^{3a-b} \end{matrix} \longrightarrow R \longrightarrow R/I_{Z_t} \longrightarrow 0 \qquad \text{if } m(a) \leq t \leq n(a), \text{ and}$$

$$0 \longrightarrow R(-a-3)^b \longrightarrow R(-a-2)^{3(b-a+1)} \longrightarrow \begin{matrix} R(-3) \\ \oplus \\ R(-a)^{3a-b} \\ \oplus \\ R(-a-1)^{3(b+1-2a)} \end{matrix} \longrightarrow R \longrightarrow R/I_{Z_t} \longrightarrow 0 \qquad \text{if } n(a) \leq t \leq P_X(a).$$

In particular, the MRC holds for Z_t.

Proof Outline

Part I: Use induction to show that the theorem holds for $m(a), n(a)$ for all a.

1. Show that the theorem holds for $a = 2, 3$ (i.e. for all $7 \leq t \leq 19$). In particular, it holds for $m(2), n(2), m(3), n(3)$.
2. For $a \geq 2$, show that $Z_{m(a)} \overset{CI(3,a,a)}{\sim} Z_{n(a)}$. It follows that the theorem is true for $m(a)$, then the theorem is true for $n(a)$.
3. For $a \geq 3$, find G_a so that $Z_{n(a-1)} \overset{G}{\sim} Z_{m(a+1)}$. Use this to get that if the theorem is true for $t = n(a-1)$, then the theorem is true for $t = m(a+1)$.
4. Induction + (1), (3), (2) gives us that the theorem is true for $m(a), n(a)$ for all a.

Part II: For $a \geq 4$, use that the theorem holds for $m(a), n(a)$ to show that it also holds for all $P_X(a-1) < t \leq P_X(a)$.

Further Details of 1

a. $7 \leq t \leq 10$: Same idea as that used to show Part II.

b. $11 \leq t \leq 19$:
   - i. Enough to show $t = 12, 15$ $(= m(a), n(a))$.
   - ii. $t = 12$: Construct one set of 12 points that has the expected resolution. Semicontinuity gives that so does a general set.
   - iii. $t = 15$: Link to Z_{12} with a complete intersection of type (3,3,3).

Further Details of 3

a. Construct a smooth aCM curve C_a containing $Z_{n(a-1)}$.

b. Find an aG set of points G_a on C_a containing $Z_{n(a-1)}$.

c. Verify that the linked set of points is a set of $m(a+1)$ points with the expected resolution.

d. Linked set is reduced and lies entirely on the smooth part of X, so by semicontinuity $Z_{m(a+1)}$ has the expected resolution.

Further Details of Part II

Idea: Let $S = k[x_0, x_1, x_2]$ and $J_{Z_t} \subset S$ be an artinian reduction of I_{Z_t}. Count linear syzygies for J_{Z_t} and the canonical module of S/J_{Z_t}.

There are three cases:

a. $P_X(a-1) < t < m(a)$

b. $m(a) < t < n(a)$

c. $n(a) < t < P_X(a)$

To show case b: $m(a) < t < n(a)$

- i. 1st module:
  - Hilbert function gives number of generators in degree a.
  - Regularity says there are no generators in degrees $> a+1$.
  - $Z_{n(a)} \supset Z_t$. So $(J_{Z_{n(a)}})_a$ generates $(J_{Z_{n(a)}})_{a+1} = (J_{Z_t})_{a+1}$ implies that so does $(J_{Z_t})_a$. So there are no generators in degree $a+1$.
- ii. For the last module, count the generators of the canonical module of S/J_{Z_t} using $Z_{m(a)} \subset Z_t$.
- iii. Middle module now follows by computation.

Cases a and c are similar.

To appear in J. Pure Appl. Algebra 215 (2011), pp. 1737-1746. Available online at: doi:10.1016/j.jpaa.2010.10.009
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