# Generalized Multiplicities and Depth of Blowup Algebras


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# Setting

- $(R, \mathfrak{m}, k)$ Cohen-Macaulay (CM) local ring.
- $|k| = \infty$, $\dim(R) = d > 0$.
- $I$ an $R$-ideal.
- $J \subseteq I$ is a minimal reduction of $I$, i.e., $I^{n+1} = JI^n$ for some $n \in \mathbb{N}$ and $J$ is minimal with respect to inclusion.
- $\ell(I)$ is the analytic spread of $I$. Recall $\mu(J) = \ell(I)$.
- $r(I) = \min\{n \,|\, I^{n+1} = JI^n,\ \text{for some minimal reduction } J\}$, the reduction number of $I$.

# MULTIPLICITIES

# Hilbert-Samuel multiplicity

If $I$ is $\mathfrak{m}$-primary,

$$e(I) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(I^n/I^{n+1})$$
$$= \lim_{n\to\infty} \frac{d!}{n^d} \lambda_R(R/I^n)$$

is the Hilbert-Samuel multiplicity of $I$.

If $I$ is not $\mathfrak{m}$-primary, then

$$\lambda_R(I^n/I^{n+1}) = \infty, \text{ and } \lambda_R(R/I^n) = \infty$$

.

# Generalized multiplicities

We use

$$\mathrm{H}^0_{\mathfrak{m}}(M) = 0 :_M \mathfrak{m}^\infty,$$

the $0th$-local cohomology of the $R$-module $M$, the largest finite length submodule of $M$.

We obtain:

$$j(I) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \lambda_R\big(\mathrm{H}^0_{\mathfrak{m}}(I^n/I^{n+1})\big),$$

the $j$-muliplicity of $I$ (Achilles-Maneresi, 1993).

$$\varepsilon(I) = \limsup_{n\to\infty} \frac{d!}{n^d} \lambda_R\big(\mathrm{H}^0_{\mathfrak{m}}(R/I^n)\big),$$

the $\varepsilon$-muliplicity of $I$ (Ulrich-Validashti, 2011).

The limit exists when $R$ is analytically unramified (Cutkosky, 2014).

# Some properties

1. $j(I) \in \mathbb{Z}_{\geqslant 0}$.
2. $\varepsilon(I)$ can be irrational (Cutkosky-Hà-Srinivasan-Theodorescu, 2005).
3. $j(I) > 0 \Leftrightarrow \varepsilon(I) > 0 \Leftrightarrow \ell(I) = d$.
4. $\varepsilon(I) \leqslant j(I)$.
5. If $I$ is $\mathfrak{m}$-primary $\Rightarrow j(I) = \varepsilon(I) = e(I)$.

***j*-multiplicity:**

- Intersection theory (Achilles-Manaresi, 1993).
- Numerical criterion for integral dependence (Rees' Theorem):
  If $J \subseteq I$, then

$$I \subseteq \overline{J} \Leftrightarrow j(I_{\mathfrak{p}}) = j(J_{\mathfrak{p}}),\ \forall \mathfrak{p} \in \operatorname{Spec} R \text{ (Flenner-Manaresi, 2001).}$$

- Conditions for Cohen-Macaulayness of blowup algebras (Polini-Xie, 2013), (Mantero-Xie, 2014), (M, 2015).

**$\varepsilon$-multiplicity:**

- Rees' Theorem for ideals and modules (Ulrich-Validashti, 2011).
- Equisingularity Theory (Kleiman-Ulrich-Validashti).

# COMPUTING GENERALIZED MULTIPLICITIES

(with Jack Jeffries and Matteo Varbaro)

Despite their importance, the generalized multiplicities are not easy to compute.

The following formula expresses the $j$-multiplicity as the length of a module:

$$j(I) = \lambda_R \left( \frac{R}{(a_1, a_2, \ldots, a_{d-1}) : I^\infty + (a_d)} \right)$$

for $a_1$, $a_2$, ..., $a_d$ general elements in $I$. (Achilles-Manaresi 1993, Xie 2012)

The $\varepsilon$-multiplicity has a better behavior than the $j$-multiplicity in some aspects, but it is harder to compute.

**Goal:** Compute generalized multiplicities for large classes of ideals.

Let $R = k[x_1, x_2, \ldots, x_d]$ and let $I$ be a monomial ideal minimally generated by $\mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{v}_n}$, where $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} \cdots x_d^{v_d}$ if $\mathbf{v} = (v_1, \ldots, v_d)$.

The Newton polyhedron of $I$ is defined to be the following convex region:

$$\mathrm{conv}(I) := \mathrm{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n) + \mathbb{R}^d_{\geqslant 0}.$$

We have $\mathbf{x}^{\mathbf{v}} \in \overline{I}$ if and only if $\mathbf{v} \in \mathrm{conv}(I)$.

Assume $I$ is $\mathfrak{m}$-primary (i.e., $I$ contains pure powers on each variable). Then $\mathrm{covol}(I) := \mathrm{vol}\big(\mathbb{R}^d_{\geqslant 0} \setminus \mathrm{conv}(I)\big)$ is finite.

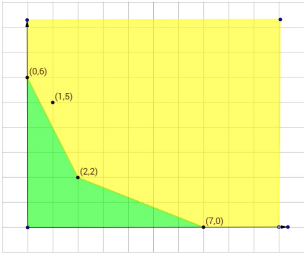
**Theorem (Teissier, 1988)**

*Let $I$ be an $\mathfrak{m}$-primary monomial ideal, then*

$$e(I) = d!\,\mathrm{covol}(I).$$

# Example

The following picture corresponds to the ideal $I = (x^7, x^2y^2, xy^5, y^6)$.




conv($I$) is the yellow region., and covol($I$) is the volume of the green region.

$$
\begin{aligned}
e(I) &= 2!\,\mathrm{covol}(I) \\
&= 26.
\end{aligned}
$$

Let $I$ be an arbitrary monomial ideal (not necessarily $\mathfrak{m}$-primary).

If $\{P_1, \ldots, P_b\}$ are the bounded faces of dimension $d-1$ of $\operatorname{conv}(I)$, we call the region

$$\operatorname{pyr}(I) = \bigcup_{i=1}^{b} \operatorname{conv}(P_i, 0),$$

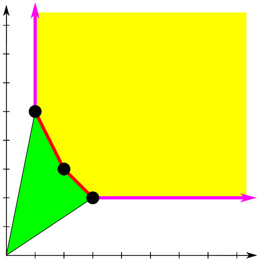
the pyramid of $I$.

**Theorem (Jeffries-M, 2013)**

$$j(I) = d! \operatorname{vol}\big(\operatorname{pyr}(I)\big).$$

# Example

The following picture corresponds to the ideal $I = (xy^5, x^2y^3, x^3y^2)$.



conv($I$) is the yellow region., and pyr($I$) is the green region.

$$\begin{aligned} j(I) &= 2!\mathrm{vol}\big(\mathrm{pyr}(I)\big) \\ &= 6 \end{aligned}$$

# The $\varepsilon$-multiplicity of monomial ideals

Let $H_i = \{x \in \mathbb{R}^n \mid \langle x, b_i \rangle = c_i\}$, with $b_i \in \mathbb{Q}^d$, $c_i \in \mathbb{Q}$ for $i = 1, \ldots, w$ be the supporting hyperplanes of $\operatorname{conv}(I)$ such that

$$\operatorname{conv}(I) = H_1^+ \cap H_2^+ \cap \cdots \cap H_w^+ .$$

Assume that $H_1, \ldots, H_u$, are the hyperplanes corresponding to unbounded facets and define

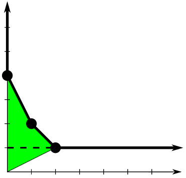
$$\operatorname{out}(I) = (H_1^+ \cap \cdots \cap H_u^+) \cap (H_{u+1}^- \cup \cdots \cup H_w^-).$$

**Theorem (Jeffries-Montaño, 2013)**

$$\varepsilon(I) = d! \operatorname{vol}\big(\operatorname{out}(I)\big).$$

# Example

Let following picture corresponds to the ideal $I = (y^4, x^2y, xy^2)$.



pyr($I$) is the green region and out($I$) is the portion of the green region that lies above the dotted line.

$$j(I) = 2!\,\mathrm{vol}\big(\mathrm{pyr}(I)\big) \qquad\qquad \varepsilon(I) = 2!\,\mathrm{vol}\big(\mathrm{out}(I)\big)$$
$$= 7 \qquad\qquad\qquad\qquad = 5$$

Notice covol($I$), vol$\big(\mathrm{pyr}(I)\big)$, and, vol$\big(\mathrm{out}(I)\big)$ coincide when $I$ is $\mathfrak{m}$-primary. Therefore, our theorems are generalizations of Teissier's theorem.

# Determinantal ideals

Consider the following matrix in $m \cdot n$ different variables $\{x_{i,j}\}$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, and $m \leqslant n$.

$$A = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{pmatrix}$$

Let $I_t$ for $t \leqslant m$ be the ideal of the polynomial ring $R = k[\{x_{i,j}\}]$ generated by all the $t$-minors of $A$.

**Theorem (Jeffries-M-Varbaro, 2015)**

*Let*

$$c = \frac{(mn-1)!}{(n-1)!(n-2)!\cdots(n-m)!\cdot m!(m-1)!\cdots 1!}.$$

*Then,*

(i)

$$j(I_t) = ct \int\limits_{\substack{[0,1]^m \\ \sum z = t}} (z_1\cdots z_m)^{n-m} \prod_{1\leqslant i<j\leqslant m} (z_j - z_i)^2 \, \mathrm{d}\nu\,;$$

(ii)

$$\varepsilon(I_t) = cmn \int\limits_{\substack{[0,1]^m \\ \max_i\{z_i\}+t-1\leqslant\sum z\leqslant t}} (z_1\cdots z_m)^{n-m} \prod_{1\leqslant i<j\leqslant m} (z_j - z_i)^2 \, \mathrm{d}z\,;$$

These integrals can be computed using the package NmzIntegrate of Normaliz.

# Rational normal scrolls

Consider positive integers $a_1 \leqslant \cdots \leqslant a_r$, and set $N = \sum_{i=1}^{r} a_i + r - 1$. The rational normal scroll associated to this sequence is the projective subvariety of $\mathbb{P}^N$, defined by the ideal

$$I = I(a_1, \ldots, a_r) \subseteq K[\{x_{i,j}\}_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant a_i + 1}]$$

generated by the 2-minors of the matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,a_1} & \cdots & x_{r,1} & x_{r,2} & \cdots & x_{r,a_r} \\ x_{1,2} & x_{1,3} & \cdots & x_{1,a_1+1} & \cdots & x_{r,2} & x_{r,3} & \cdots & x_{r,a_r+1} \end{pmatrix}.$$

# The $j$-multiplicity of rational normal scrolls

**Theorem (Jeffries-M-Varbaro, 2015)**

$j(I(a_1, \dots, a_r)) =$

$$\begin{cases} 0 & \text{if } c < r+3\,, \\ 2 \cdot \left( \binom{2c-4}{c-2} - \binom{2c-4}{c-1} \right) & \text{if } c = r+3\,, \\ 2 \cdot \left( \sum_{j=2}^{c-r-1} \binom{c+r-1}{c-j} - \binom{c+r-1}{c-1}(c-r-2) \right) & \text{if } c > r+3\,. \end{cases}$$

### Example

- $$I(4) = I_2 \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}$$
  Here $c = 4$ and $r = 1$, $c = r + 3$ therefore
  $$j(I(4)) = 2 \cdot \left( \binom{4}{2} - \binom{4}{3} \right) = 4.$$

- $$j(I(3,2)) = j\left( I_2 \begin{pmatrix} x_1 & x_2 & x_3 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_6 & x_7 \end{pmatrix} \right) = 10\,.$$

These examples had been computed by Nishida-Ulrich in 2010 using residual intersection theory and some intricate computations.

# Binomial ideals?

Binomial ideals form another class of ideals with combinatorial structure.

**Problem:** Compute the generalized multiplicities of binomial ideals.

One may consider first ideals $I$ defining numerical semigroup rings, i.e.,

$$k[[x_1, \ldots, x_d]]/I \cong k[[t^{a_1}, \ldots, t^{a_d}]]$$

for some positive integers $a_1 < \cdots < a_d$.

We know $\ell(I) = d$ if $I$ is not a complete intersection (Cowsik-Nori, 1976). Hence $j(I) \neq 0$.

Nishida-Ulrich gave a explicit formula for $j(I)$ in the case $d = 3$.

# MINIMAL MULTIPLICITIES AND DEPTH OF BLOWUP ALGEBRAS

# Blowup algebras

- $\mathcal{R}(I) = R[It] = \bigoplus_{n\geqslant 0} I^n t^n$, the Rees algebra of $I$.
- $\mathcal{G}(I) = \bigoplus_{n\geqslant 0} I^n/I^{n+1}$, the associated graded algebra of $I$.
- $\mathcal{F}(I) = \bigoplus_{n\geqslant 0} I^n/\mathfrak{m}I^n$, the fiber cone of $I$.

If $S$ is any of these algebras, then depth $S := \operatorname{depth}_{\mathfrak{M}} S$, where $\mathfrak{M} = \mathfrak{m} + \mathcal{R}(I)_+$.

- $\dim \mathcal{R}(I) = d + 1$, provided ht $I > 0$.
- $\dim \mathcal{G}(I) = d$.
- $\dim \mathcal{F}(I) = \ell(I)$.

$S$ is CM if depth $S = \dim S$.

# Depths of blowup algebras

**Question:** How do the depths of the blowup algebras relate to each other?

Assume ht $I \geqslant 1$.

- $\mathcal{R}(I)$ is CM $\Rightarrow \mathcal{G}(I)$ is CM. (Huneke, 1982)
- "$\Leftarrow$" if $\sqrt{I} = \mathfrak{m}$ and $r(I) < d$. (Goto-Shimoda, 1982)
- "$\Leftarrow$" if $a(\mathcal{G}(I)) < 0$. (Ikeda-Trung, 1989)
- "$\Leftarrow$" if $R$ regular. (Lipman, 1994)
- $\mathcal{G}(I)$ is not CM $\Rightarrow$ depth $\mathcal{R}(I) =$ depth $\mathcal{G}(I) + 1$ (Huckaba-Marly, 1994)

However, in general:

- $\mathcal{F}(I)$ is CM $\not\Rightarrow \mathcal{G}(I)$ is CM.
- $\mathcal{F}(I)$ is CM $\not\Leftarrow \mathcal{R}(I)$ is CM.

# Minimal multiplicities

Notions of minimal multiplicity provide conditions for strong relations among the depths of blowup algebras. They originated in Abhyankar's inequality

$$e(\mathfrak{m}) \geqslant \mu(\mathfrak{m}) - d + 1.$$

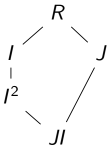
- $R$ is of minimal multiplicity $\Rightarrow \mathcal{G}(\mathfrak{m})$ is CM (Sally, 77')

**Sally's conjecture:**

- $R$ is of almost minimal multiplicity $\Rightarrow \operatorname{depth} \mathcal{G}(\mathfrak{m}) \geqslant d - 1$. (Rossi-Valla, 96', Wang, 97')

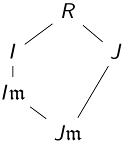
# Minimal multiplicities

$I$ $\mathfrak{m}$-primary. Recall, $e(I) = e(J) = \lambda(R/J)$.




$e(I) \geqslant \lambda(I/I^2) - (d-1)\lambda(R/I)$ with equality iff $r(I) \leqslant 1$.

- $I$ is of minimal multiplicity $\Rightarrow \mathcal{G}(I)$ is CM. (Valla, 1978)
  $\Rightarrow \mathcal{F}(I)$ is CM. (Huneke-Sally, 1988)
- $I$ is of almost minimal multiplicity $\Rightarrow$ depth $\mathcal{G}(I) \geqslant d-1$. (Rossi, 2000)




$e(I) \geqslant \mu(I) - d + \lambda(R/I)$ with equality iff $I\mathfrak{m} = J\mathfrak{m}$.

- $I$ is of Goto-minimal multiplicity:
  $\mathcal{R}(I)$ is CM $\Leftrightarrow \mathcal{G}(I)$ is CM $\Leftrightarrow r(I) \leqslant 1$. (Goto, 2000)
- $I$ is of almost Goto-minimal multiplicity:
  depth $\mathcal{G}(I) \geqslant d-2 \Rightarrow$ depth $\mathcal{F}(I) \geqslant d-1$.
  (Jayanthan-Verma, 2005)

**Question:** How can we define notions of minimal multiplicities for non-$\mathfrak{m}$-primary ideals?

**Goal:** Use the $j$-multiplicity to extend properties of minimal multiplicity to non-$\mathfrak{m}$-primary ideals.

**Theorem (Achilles-Manaresi, Xie)**

*Let $x_1, \ldots, x_{d-1}$ be $d-1$ general elements in $I$ and $\widetilde{R} := R/(x_1, \ldots, x_{d-1}) : I^\infty$. The ideal $\widetilde{I} := I\widetilde{R}$ is $\widetilde{\mathfrak{m}}$-primary and $j(I) = e(\widetilde{I})$.*

Therefore,

$$j(I) \geqslant \lambda(\widetilde{I}/\widetilde{I}^2) \qquad \text{and} \qquad j(I) \geqslant \mu(\widetilde{I}) - 1 + \lambda(\widetilde{R}/\widetilde{I}).$$

# Minimal $j$-multiplicity

Polini and Xie in defined $I$ to be of minimal $j$-multiplcity if $\ell = d$ and

$$j(I) \underset{\geqslant}{=} \lambda(\widetilde{I}/\widetilde{I^2}),$$

and they extended the results of Rossi, Valla, and Wang. They proved:

**Theorem (Polini-Xie, 2013)**

*Assume* $\operatorname{depth}(R/I) \geqslant \min\{\dim(R/I), 1\}$*, and $I$ satisfies $G_d$ and $AN^-_{d-2}$. If $I$ is of minimal j-multiplicity, then $\mathcal{G}(I)$ is Cohen-Macaulay.*

**Theorem (Polini-Xie, 2013)**

*Assume* $\operatorname{depth}(R/I) \geqslant \min\{\dim(R/I), 1\}$*, and $I$ satisfies $G_d$ and $AN^-_{d-2}$. If $I$ is of almost minimal j-multiplicity, then* $\operatorname{depth}\mathcal{G}(I) \geqslant d-1$.

# Discussion: Residual assumptions

When $I$ is $\mathfrak{m}$-primary, one is able to reduce to lower dimensions by modding out regular sequences in $I$.

If $I$ is not $\mathfrak{m}$-primary, the lack of the above property is a big complication if one desires to generalize results that hold in the $\mathfrak{m}$-primary case.

Some residual assumptions are necessary in one would like to proceed by induction on the dimension of the ambient ring.

We use Artin-Nagata properties!

# Artin-Nagata properties: Examples

Set $\ell := \ell(I)$.

$I$ satisfies $G_\ell$ if $\mu(I_\mathfrak{p}) \leqslant \operatorname{ht}\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$ such that $\operatorname{ht}\mathfrak{p} < \ell$.

Classes of ideals satisfying $G_\ell$ and $AN^{-}_{\ell-2}$:

- $\ell = \operatorname{ht}(I)$, i.e., equimultiple ideals.
- One dimensional ideals that are generically complete intersection.
- Cohen-Macaulay ideals generated by $n \leqslant \operatorname{ht} I + 2$ elements satisfying $G_\ell$. (Avramov-Herzog)
- Perfect ideals of height two satisfying $G_\ell$. (Apéry, Huneke)
- Perfect Gorenstein ideals of height three satisfying $G_\ell$. (Watanabe, Huneke)
- Initial lex-segment ideals (Smith, Fouli-M).

Back to multiplicities...
We define $I$ to be of Goto-minimal $j$-multiplcity if $\ell = d$ and

$$j(I) \underset{\geqslant}{=} \mu(\widetilde{I}) - 1 + \lambda(\widetilde{R}/\widetilde{I}).$$

**Proposition (M, 2015)**

*Assume $I$ satisfies $G_d$ and $AN^-_{d-2}$, then*

*$I$ is of Goto-minimal j-multiplicity $\Leftrightarrow I\mathfrak{m} = J\mathfrak{m}$ for one (hence every) minimal reduction $J$ of $I$.*

This proposition is a consequence of Corso-Polini-Ulrich formula for the core.

# Theorem 1

From now on, we will assume assume $I$ satisfies $G_\ell$ and $AN^-_{\ell-2}$.

**Theorem (M, 2015)**

*Assume $J \cap I^n\mathfrak{m} = JI^{n-1}\mathfrak{m}$ for every $2 \leqslant n \leqslant r(I)$, then TFAE:*

(i) *$\mathcal{F}(I)$ is CM.*

(ii) $\operatorname{depth} \mathcal{G}(I) \geqslant \ell - 1$.

(iii) $\operatorname{depth} \mathcal{R}(I) \geqslant \ell$.

The following corollary recovers results of Shah and Cortadellas-Zarzuela.

**Corollary**

*If $r(I) \leqslant 1$ then $\mathcal{F}(I)$ is CM.*

# Theorem 2

$h = \mathrm{ht}\,(I)$.

**Theorem (M, 2015)**

*Assume $I\mathfrak{m} = J\mathfrak{m}$, consider the following statements:*

(i) $\mathcal{R}(I)$ *is CM,*

(ii) $\mathcal{G}(I)$ *is CM,*

(iii) $\mathcal{F}(I)$ *is CM and* $a(\mathcal{F}(I)) \leqslant -h+1$,

(iv) $r(I) \leqslant \ell - h + 1$.

*Then* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

*If in addition* depth $R/I^j \geqslant d - h - j + 1$ *for every* $1 \leqslant j \leqslant \ell - h + 1$, *then all the statements are equivalent.*

1. The monomial ideals

$$I = (x_1^2, x_1x_2, \ldots, x_1x_d, x_2^2, x_2x_3, \ldots, x_2x_n)$$

for $n \geqslant 3$ in the ring $k[[x_1, \ldots, x_d]]$ are lex-segment ideals of height 2, and satisfy $G_d$ and $AN_{d-2}^-$. $I$ is of Goto-minimal $j$-multiplicity and $\mathcal{G}(I)$ is CM, then by Theorem 1 the algebras $\mathcal{R}(I)$ and $\mathcal{F}(I)$ are CM as well.

2. Let $R = k[[x, y, z, w]]$ and

$$M = \begin{pmatrix} x & y & z & w \\ w & x & y & z \end{pmatrix}.$$

The ideal $I = I_2(M)$ is CM with $h = 3$, $\ell = 4$, and satisfies $G_4$ and $AN_2^-$. I is of Goto-minimal $j$-multiplicity and $r(I) \leqslant 2$, then by Theorem 2 the algebras $\mathcal{R}(I)$, $\mathcal{G}(I)$, and $\mathcal{F}(I)$ are CM.

Thank you!

# References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