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3.2 Properties of Division

* Given 2 polynomials $f(x)$ and $p(x)$ with $p(x) \neq 0$ there exist unique $q(x)$ and $r(x)$ such that

$$f(x) = p(x) \underbrace{\boxed{q(x)}}_{\text{quotient}} + \underbrace{\boxed{r(x)}}_{\text{remainder}}$$

with degree $r(x) <$ degree $p(x)$

* Note that if $p(x) = x - c$ then

$$f(x) = (x-c) \cdot q(x) + \underbrace{\boxed{\text{constant}}}_{\text{has degree } 0}$$

and after evaluation at $x = c$ we discover that

$$f(c) = (c-c)\, q(c) + \text{constant}$$

Remainder Theorem

$$\therefore \text{constant} = f(c)$$

I.e. $$\boxed{f(x) = (x-c)\, q(x) + f(c)}$$

* Use the division algorithm to find the quotient and remainder if

$f(x) = 3x^4 + 2x^3 - x^2 - x - 6 \qquad p(x) = x^2 + 1$

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$$
\begin{array}{r|l}
 & 3x^2+2x-4 \\
\hline
x^2+1 & 3x^4+2x^3-x^2-x-6 \\
 & 3x^4 \qquad\quad +3x^2 \\
\hline
 & \qquad 2x^3-4x^2 \\
 & \qquad 2x^3 \qquad\quad +2x \\
\hline
 & \qquad\qquad -4x^2-3x \\
 & \qquad\qquad -4x^2 \qquad\quad -4 \\
\hline
 & \qquad\qquad\qquad -3x-2
\end{array}
$$

$\therefore\ q(x) = 3x^2+2x-4 \qquad r(x) = -3x-2$

Use the remainder theorem to find $f(c)$

if $f(x) = 2x^3+4x^2-3x-1 \qquad c = 2$

We need to divide by $x-2$

$$
\begin{array}{r|l}
 & 2x^2+8x+13 \\
\hline
x-2 & 2x^3+4x^2-3x-1 \\
 & 2x^3-4x^2 \\
\hline
 & \qquad 8x^2 \\
 & \qquad 8x^2-16x \\
\hline
 & \qquad\qquad 13x \\
 & \qquad\qquad 13x-26 \\
\hline
 & \qquad\qquad\qquad 25
\end{array}
$$

$\therefore\ f(2) = \text{remainder} = \underline{25}$

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* Do the same problem using the synthetic division algorithm (or Ruffini's Method)

$$
c \longrightarrow 2 \;\; \begin{array}{|cccc|c}
 & 2 & 4 & -3 & -1 \\
 & & 4 & 16 & 26 \\
\hline
 & 2 & 8 & 13 & 25
\end{array}
$$

* Show that $c = -2$ is a zero of the polynomial $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4$

$$
\begin{array}{r|cccc|c}
-2 & 3 & 8 & -2 & -10 & 4 \\
 & & -6 & -4 & 12 & -4 \\
\hline
 & 3 & 2 & -6 & 2 & 0
\end{array}
$$

$\therefore \; f(-2) = 0$

* Divide $f(x) = -5x^2 + 3$ by $p(x) = x^3 - 3x + 9$

Well :

$$-5x^2 + 3 = \underset{\substack{\uparrow \\ \text{quotient}}}{0} \cdot (x^3 - 3x + 9) + \underset{\substack{\uparrow \\ \text{remainder}}}{(-5x^2 + 3)}$$

There is nothing to do if degree f < degree $p(x)$

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Notice that the factor theorem says that from the equation

$$f(x) = (x-c)\,g(x) + f(c)$$

it follows that $(x-c)$ is a factor of $f(x)$ $\iff$ $f(c) = 0$.

* Show that $x-y$ is a factor of $x^n - y^n$ for every n.

Ans: $f(x) = x^n - y^n$ Now, we conclude from the Factor Theorem since $f(y) = y^n - y^n = 0$

Thus $x-y$ divides $x^n - y^n$.

* Show that $x-c$ is not a factor of $f(x) = 3x^4 + x^2 + 5$ for any real # c.

$f(c) = 3c^4 + c^2 + 5 > 0$ always !!