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3.3 Zeros of Polynomials

The zeros of a polynomial $f(x)$ are the solutions of the equation

$$f(x) = 0$$

Observe that each "real" zero corresponds to the x-intercept of the graph of f.

* Fundamental Theorem of Algebra

(by Friedrich Gauss, 1777-1855)

If $f(x)$ is a polynomial of positive degree and complex coefficients then $f(x)$ has at least one complex zero.

I.e. $$f(x) = (x - \underset{\mathbb{C}}{\underset{\cap}{c}})\, g(x)$$

* Thus, at least in theory we conclude that the above f can be factored as

$$f(x) = a\,(x-c_1)(x-c_2)\cdots(x-c_n)$$

each $c_k \in \mathbb{C}$ is a zero of $f(x)$.

* Notice that a polynomial of degree n has <u>at most n</u> different zeros.

Indeed, some zeros could be repeated

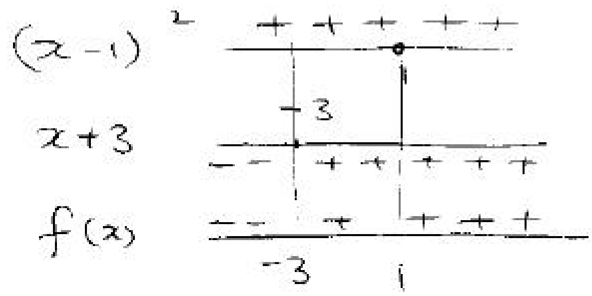
<u>Ex</u>: $f(x) = x^3 + x^2 - 5x + 3$

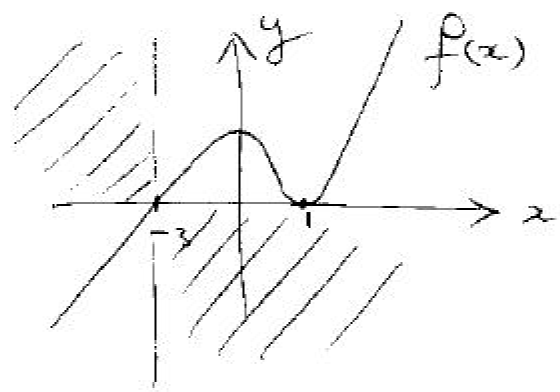
$$= (x+3)(x-1)^2$$

$x = 1$ is a zero of multiplicity 2

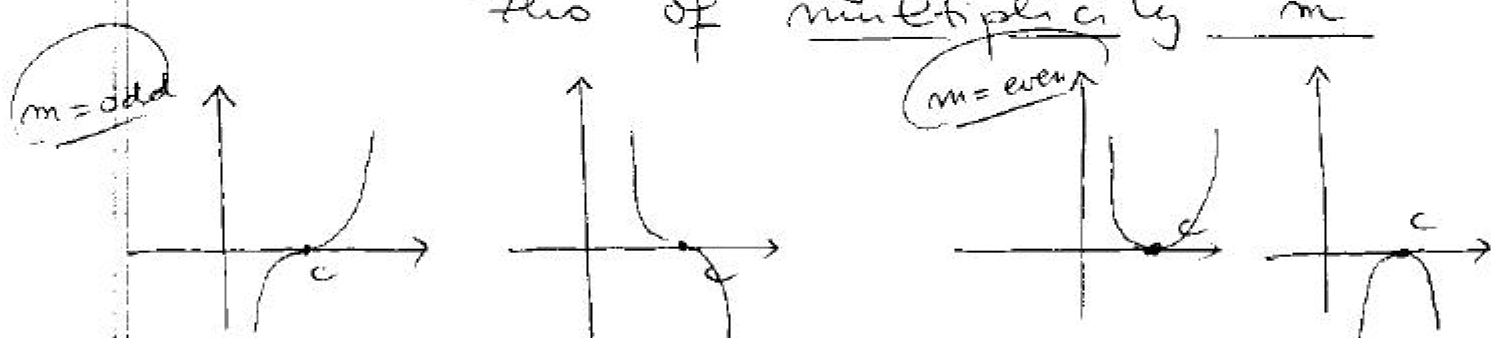
$x = -3$ is a simple zero

the graph of $f(x)$ looks like:





<u>In general</u> if there is a factor $(x-c)^m$ in $f(x)$ we say that c is a zero of <u>multiplicity m</u>



<u>general shape of graph near $x=c$</u>

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Descartes' Rule of Signs

Let $f(x)$ be a polynomial with real coefficients and a non-zero constant term:

(1) The number of positive real zeros of $f(x)$ either is equal to the number of variations of sign in $f(x)$ or is less than that number by an even number.

(2) The number of negative real zeros of $f(x)$ is either equal to the number of variations of sign in $f(-x)$ or is less than that number by an even number.

Ex: $$f(x) = 2\underbrace{x^5 - 7}_{+\text{ to }-}\underbrace{x^4 + 3}_{-\text{ to }+}x^2 + 6\underbrace{x - 5}_{+\text{ to }-}$$

3 variations

$$f(-x) = -2x^5 - 7\underbrace{x^4 + 3}_{-\text{ to }+}\underbrace{x^2 - 6}_{+\text{ to }-2}x - 5$$

2 variations

Possibilities

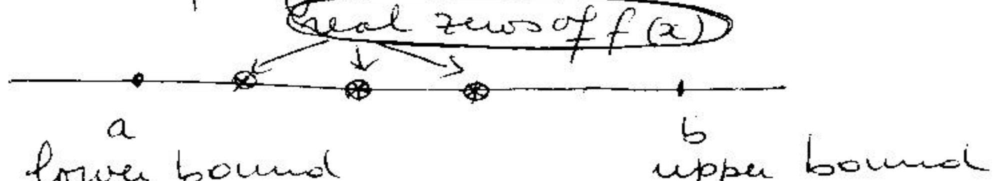
# of positive real zeros	3	3	1	1
# of negative real zeros	2	0	2	0
# of imaginary complex zeros	0	2	2	4
TOTAL # of zeros	5	5	5	5

<u>Note</u> : the constant term has to be $\neq 0$
<u>Otherwise</u> write $f(x)$ as

$$f(x) = x^n \cdot \underbrace{g(x)}$$

where $g(x)$ has $\neq 0$ constant term.

* We would like now to find explicit upper and lower bounds for the real zeros of a polynomial $f(x)$:



$\boxed{\text{Theorem 1}}$

Suppose $f(x)$ has real coefficients and positive leading term. Suppose that $f(x)$ is <u>sinthetycally</u> divided by $x - c$

(1) If $c > 0$ and if all #'s in the third row of the division process are either positive or zero, then c is an upper bound for the real zeros of f

(2) If $c < 0$ and if all #'s in the third rows are alternatively positive or negative then c is a lower bound for the real zeros of f.

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Example:

Let $f(x) = 2x^3 + 5x^2 - 8x - 7$

Consider Descartes' rule of signs

$$f(x) = 2x^3 + \underbrace{5x^2 - 8x}_{+ \text{ to } -} - 7$$

$$f(-x) = \underbrace{-2x^3 + 5x^2}_{- \text{ to } +} \underbrace{+ 8x - 7}_{+ \text{ to } -}$$

Thus $f(x)$ has 1 positive real zero and either 2 or 0 negative zeros.

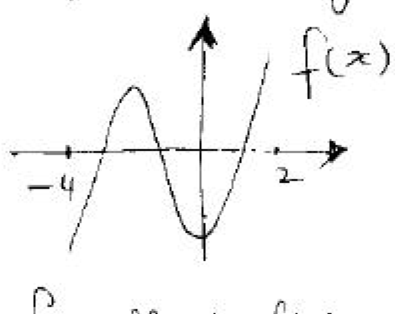
1	2	5	−8	−7
		2	7	−1
	2	7	−1	−8

2	2	5	−8	−7
		4	18	20
	2	9	10	13

Thus $\underline{c = 2}$ is an upper bound for the positive real zeros of f

$f(x) = (x-2)(x^2 + 9x + 10) + 13$ i.e. for values greater than or equal to 2 $f(x)$ is always positive !!

−4	2	5	−8	−7
		−8	+12	−16
	2	−3	4	−23



$f(x) = (x+4)(2x^2 - 3x + 4) - 23$ i.e. for all values smaller than or equal to -4 $f(x)$ is always negative!

Theorem 2

Suppose $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

All zeros of f are in the symmetric interval $(-M, M)$

where
$$M = \frac{\max\left(|a_n|, |a_{n-1}|, \dots, |a_1|, |a_0|\right)}{|a_n|} + 1$$

Ex: in the example of before

$$f(x) = 2x^3 + 5x^2 - 8x - 7$$

$$M = \frac{\max(2, 5, 8, 7)}{2} + 1 = \frac{8}{2} + 1 = 5$$

Thus the zeros of f are inside

$$(-5, 5)$$

Notice that this result is easier to verify in practice. That's why the bound is less sharp!!