4.1 Inverse functions

Our goal in this chapter is to introduce two new functions not of polynomial type:

$y = a^x$ exponential functions ⇄ logarithmic function $y = \log_a x$

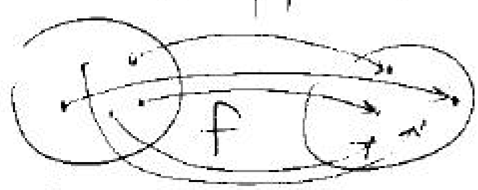
and they are inverses of each other.

We say, in general, that f <u>is one-one</u> (or <u>injective</u>) if any of the following properties are satisfied:

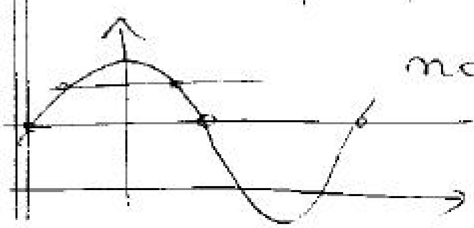
(1) $a \neq b \Longrightarrow f(a) \neq f(b)$

(2) $f(a) = f(b) \Longrightarrow a = b$

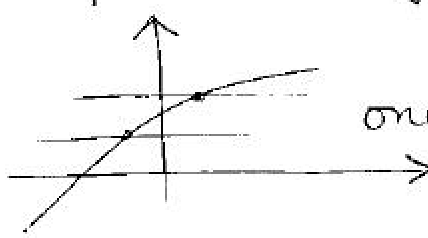
This means that different elements are taken by f into different elements



For real valued functions, we can decide if f is one-one from its graph



not one-one

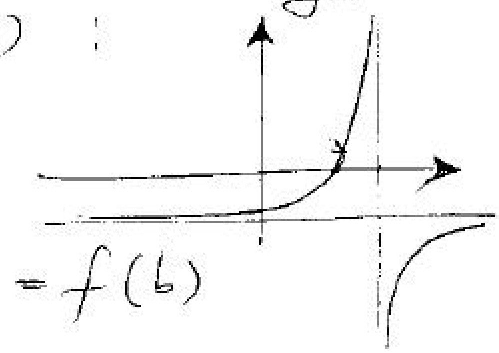


one-one

vii) **Horizontal line test:**

If any horizontal line meets the graph of f in at most one point then f is one-one.

<u>Ex:</u> Let's check that f is one-one using the second definition (2):



$$f(x) = \frac{3-x}{x-4}$$

$$f(a) = \frac{3-a}{a-4} \overset{\text{suppose}}{=} \frac{3-b}{b-4} = f(b)$$

$$(3-a)(b-4) = (3-b)(a-4)$$

$$3b - \cancel{12} - \cancel{ab} + 4a = 3a - \cancel{12} - \cancel{ab} + 4b$$

$$3b - 4b = 3a - 4a$$

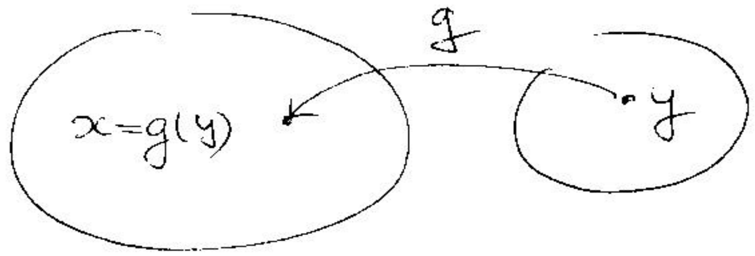
$$-b = -a \quad \text{or} \quad b = a \;!!$$

<u>In general</u>: increasing or decreasing functions are one-one.

We can reverse the correspondence in the case of one-one functions

$$x \xrightarrow{f} y = f(x)$$

We can define a function g



g is the inverse of f and it is unique!
It is denoted by f^{-1}.

It has the properties that:

$$(*) \quad g(f(x)) = g(y) = x \qquad [f^{-1} \circ f = id_X]$$

$$* \quad f(g(y)) = f(x) = y \qquad [f \circ f^{-1} = id_Y]$$

Notice that domains and ranges for f and $g = f^{-1}$ are reversed!

Warning: perhaps we rather be able to graph f and $g = f^{-1}$ in the same Cartesian plane using the same x-variable. Thus we need to invert the role of y and x in f^{-1}.

Ex: find the inverse of $y = \frac{3-x}{x-4}$

$$y(x-4) = 3-x \qquad xy + x = 3+4y$$

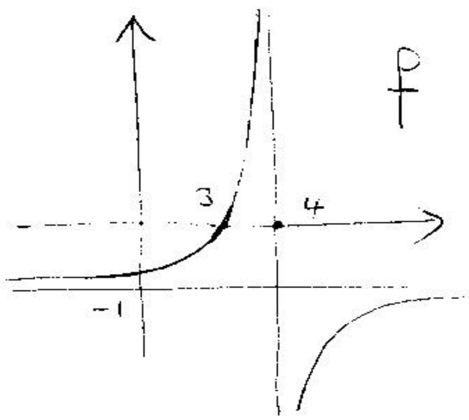
$$x(1+y) = 3+4y$$

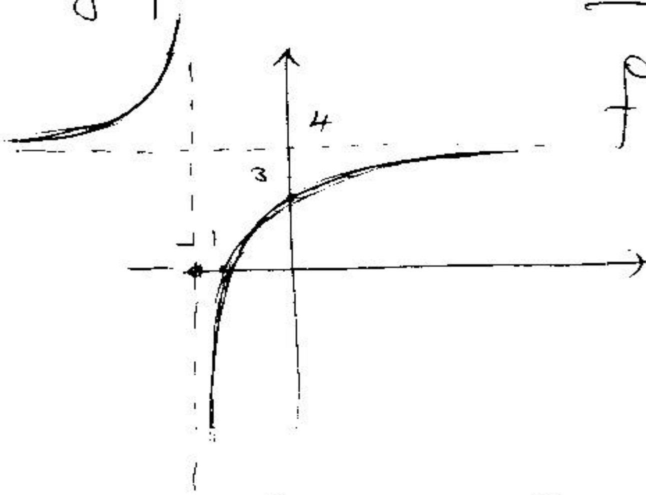
$$\therefore \quad x = \frac{3+4y}{1+y} \qquad \underline{\text{Now}} \text{ invert } x \longleftrightarrow y$$

$$y = \frac{3+4x}{1+x}$$

Notice that if $f(x) = \dfrac{3-x}{x-4}$ and $g(x) = \dfrac{3+4x}{1+x}$

then $(f \circ g)(x) = x \quad (g \circ f)(x) = x$. <u>CHECK!</u>



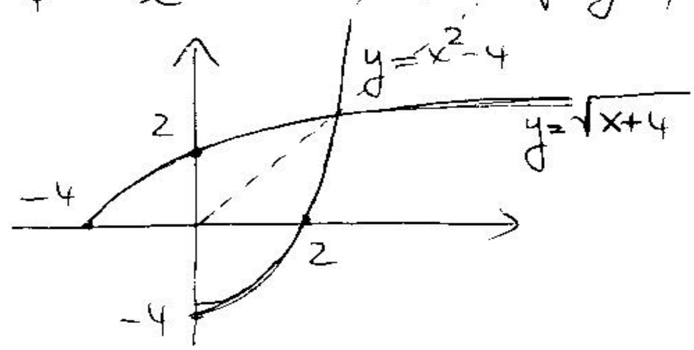


$f^{-1} = g$

<u>Ex</u>: Find the inverse of $y = x^2 - 4$ for $x \geq 0$.

$y = x^2 - 4 \quad \leadsto \quad y + 4 = x^2 \longrightarrow x = \sqrt{y+4}$

$y = f^{-1}(x) = \sqrt{x+4}$



<u>EFFECT ON GRAPHS</u> :

The graphs of f and f^{-1} are symmetric with respect to the line $\underline{\underline{y = x}}$!