4.3 The Natural Exponential Function

We discussed — in the previous lecture — that the amount of money deposited in a bank grows according to the formula

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt} \quad \text{where}$$

n = # of interest periods per year

r = interest rate

t = time in years

If n grows of course we expect better profit

Example: $P_0 = \$1{,}000$ $\quad r = 9\%$ or 0.09

* if $n=4$ (i.e. the interest is computed each quarter)
$P(1) = 1{,}000\left(1 + \frac{0.09}{4}\right)^4 = \$1{,}093.08$

* if $n=12$ (i.e. the interest is computed each month)
$P(1) = 1{,}000\left(1 + \frac{0.09}{12}\right)^{12} = \$1{,}093.81$

* if $n=52$ (i.e. the interest is computed each week)
$P(1) = 1{,}000\left(1 + \frac{0.09}{52}\right)^{52} = \$1{,}094.09$

* if $n=365$ (i.e. the interest is computed each day)
$P(1) = 1{,}000\left(1 + \frac{0.09}{365}\right)^{365} = \$1{,}094.16$

* if $n=8760$ (i.e. the interest is computed each hour)
$P(1) = 1{,}000\left(1 + \frac{0.09}{8760}\right)^{8760} = \$1{,}094.17$

* if $n = 525{,}600$ (i.e. the interest is computed each minute)

$$P(1) = 1{,}000\left(1 + \frac{0.09}{525{,}600}\right)^{525{,}600} = \$\ 1{,}094.17$$

Thus $P(1)$ approaches a fixed value as n increases.

If $n \to \infty$ we say that the interest is <u>compounded continuously</u>.

What's the formula in this case?

$$P_0\left(1 + \frac{r}{n}\right)^{nt} = P_0\left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{rt} \xrightarrow[n \to +\infty]{} P_0\, e^{rt}$$

This is because the sequence of numbers

$$\left(1 + \frac{1}{q}\right)^{q} \xrightarrow[\text{as } q \to \infty]{\text{approach}} e \cong 2.71828 \quad \left(\text{Euler/Napier number}\right)$$

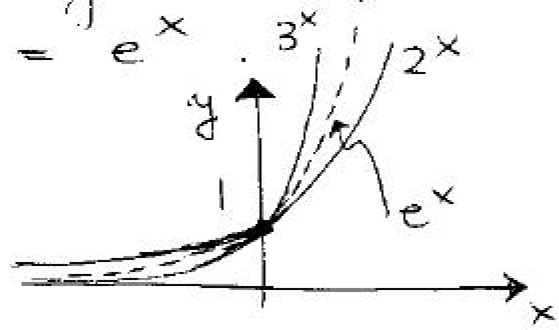
as you can convince yourself after building a chart of values

q	$q = 1$	$q = 10$	$q = 1{,}000$	$q = 1{,}000{,}000$
$\left(1 + \frac{1}{q}\right)^{q}$	2	2.59374	2.71692	2.71828

Thus: $\underline{P(t) = P_0\, e^{rt}}$ is the formula for the interest compounded continuously.

* The natural exponential function is defined by $f(x) = e^x$.

Its graph looks like



* Ex: How much money invested at the interest rate of $r = 9.5\%$ compounded continuously will amount to \$15,000 after 4 years?

$15{,}000 = P_0 e^{0.095 \cdot 4}$ $\quad\therefore P_0 = \dfrac{15{,}000}{e^{0.38}} = 15{,}000\, e^{-0.38}$

$= \$\ 10{,}257.92$

* Ex: An investment of \$400 increased to $P = \$\ 890.20$ in 16 years. Find the interest rate r if the interest was compounded continuously.

We need to solve the equation

$$890.20 = 400\, e^{4r}$$

$$e^{4r} = \frac{890.20}{400} = 2.2255$$

We can check with the calculator that $r = 0.05$ (or 5%) works.

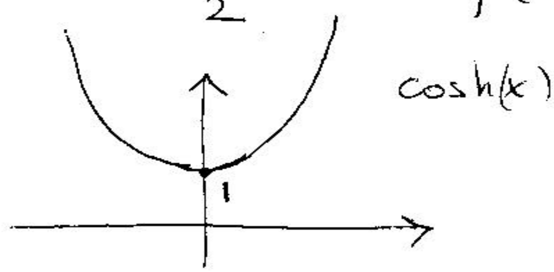
The exact answer is: $\quad r = \frac{1}{4} \ln(2.2255)$

(we'll see this later!)

Ex. The function $f(x) = \dfrac{e^{x}+e^{-x}}{2}$

is called the hyperbolic cosine function.
Observe that $f(-x) = \dfrac{e^{-x}+e^{-(-x)}}{2} = f(x)$
i.e. $f(x)$ is even

Its graph looks like



It can be shown that a uniform flexible cable hangs from 2 poles of the same height according to a shape described by

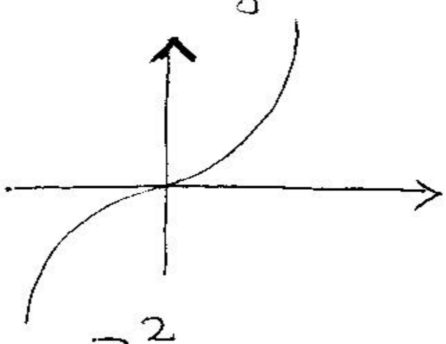
$$g(x) = \frac{a}{2}\left(e^{x/a} + e^{-x/a} \right)$$

The case $a=1 \rightsquigarrow \boxed{\cosh(x) = \dfrac{e^{x}+e^{-x}}{2}}$

Similarly the hyberbolic sine is defined by

$$\sinh(x) = \frac{e^{x}-e^{-x}}{2}$$

this function is odd!



Observe that: $[\cosh(x)]^2 - [\sinh(x)]^2 = \dots = 1$

in fact $\left(\dfrac{e^{x}+e^{-x}}{2}\right)^2 - \left(\dfrac{e^{x}-e^{-x}}{2}\right)^2 = \dfrac{e^{2x}+2+e^{-2x}-(e^{2x}-2+e^{-2x})}{4} = 1$

Ex: Find the zeros of $-x^2e^{-x}+2xe^{-x}=0$

$e^{-x}\left[2x-x^2\right]=0 \rightsquigarrow 2x-x^2=0 \rightsquigarrow x=0,2$ $\boxed{\text{as } e^{-x} \neq 0}$