4.6 Exponential and Logarithmic Equations

Base change formula: let $a, b, u > 0$

$$\log_b u = \frac{\log_a u}{\log_a b}$$

special cases: $\log_b u = \dfrac{\log u}{\log b} = \dfrac{\ln u}{\ln b}$

Indeed $y = \log_b u \iff b^y = u$. Take now $\log_a(\cdot)$ of both sides and we get

$$y \log_a(b) = \log_a(b^y) = \log_a u$$

$$\therefore \quad y = \frac{\log_a(u)}{\log_a(b)} = \log_b u.$$

* Example: evaluate $\dfrac{\log_7(243)}{\log_7(3)} =$

$$= \frac{\frac{\log(243)}{\log 7}}{\cancel{\log 3}/\log 7} = \frac{\log(243)}{\log 3} = 5 \quad (\text{calc.})$$

however you could have realized this right away because: $\log_7(243) = \log_7(3^5)$
$= 5 \log_7 3$!!

* Solve the equation : $\boxed{3^{x+4} = 2^{1-3x}}$

$(x+4)\log 3 = (1-3x)\log 2$

$x\log 3 + 4\log 3 = \log 2 - 3x\log 2$

$x\log 3 + 3x\log 2 = \log 2 - 4\log 3$

$x\left[\log 3 + \log 2^3\right] = \log\left[\frac{2}{3^4}\right]$

$x = \dfrac{\log(2/81)}{\log(24)} \cong -1.16$

* Solve the equation $\boxed{4^x - 3(4^{-x}) = 8}$

$4^x - \dfrac{3}{4^x} = 8 \iff 4^x \cdot 4^x - 3 - 8 \cdot 4^x = 0$

$(4^x)^2 - 8\cdot 4^x - 3 = 0$ <u>Set</u> $y = 4^x$ so we get

$y^2 - 8y - 3 = 0$ Use the quadratic eq:

$y = \dfrac{8 \pm \sqrt{64+12}}{2} = \dfrac{8 \pm 2\sqrt{19}}{2} = 4 \pm \sqrt{19}$

Hence $4^x = \begin{cases} 4+\sqrt{19} \\ \underbrace{4-\sqrt{19}}_{<0} \text{ impossible} \end{cases}$

$\therefore\ 4^x = 4 + \sqrt{19}$ or $x = \dfrac{\log(4+\sqrt{19})}{\log 4} \cong 1.53$

* Solve the equation

$\boxed{\log(5x+1) = 2 + \log(2x-3)}$

$$\log(5x+1) - \log(2x-3) = 2\cdot\log_{10} 10$$

$$\log\left(\frac{5x+1}{2x-3}\right) = \log 10^2 \iff \frac{5x+1}{2x-3} = 100$$

$$5x+1 = 200x - 300 \qquad x = \frac{301}{195} \cong 1.54$$

* Solve the equation : $\boxed{\log(x^3) = (\log x)^3}$

$$3\log x = (\log x)^3 \qquad \text{let } y = \log x$$

$$y^3 - 3y = 0 \qquad y(y^2-3) = 0$$

$$\therefore\ y = 0 \quad \text{or} \quad y = \sqrt{3} \quad \text{or} \quad y = -\sqrt{3}$$

Hence $\log x = 0 \iff x = 1$ OR $\log x = \sqrt{3}$ $\iff x = 10^{\sqrt{3}}$ OR $\log x = -\sqrt{3} \iff x = 10^{-\sqrt{3}}$

* Solve for x in term of y if $\boxed{y = \dfrac{10^x + 10^{-x}}{10^x - 10^{-x}}}$

$$y = \frac{10^x + \frac{1}{10^x}}{10^x - \frac{1}{10^x}} \iff y = \frac{(10^x)^2+1}{(10^x)^2-1}$$

$$(10^{2x}-1)y = 10^{2x}+1 \qquad 10^{2x}y - 10^{2x} = 1+y$$

$$10^{2x}[y-1] = 1+y \qquad 10^{2x} = \frac{1+y}{y-1}$$

$$2x\,\underbrace{\log 10}_{=1} = \log\left(\frac{1+y}{y-1}\right) \rightsquigarrow x = \frac{1}{2}\log\left(\frac{y+1}{y-1}\right)$$

* To describe the acidity or basicity of solutions chemists use a number denoted by

$$\boxed{pH = -\log [H^+]}$$

where $[H^+]$ is the hydrogen ion concentration in moles per liter.

carrots have $[H^+] \cong 1.0 \times 10^{-5} \longrightarrow pH = -\log(1.0 \times 10^{-5}) = -(-5) = 5$

A solution is considered basic if $[H^+] < 10^{-7}$ or acidic if $[H^+] > 10^{-7}$.

$(pH > 7)$ Basic: $[H^+] < 10^{-7} \Leftrightarrow pH = -\log[H^+] > -\log 10^{-7} = 7$

$(pH < 7)$ Acidic: $[H^+] > 10^{-7} \Leftrightarrow pH = -\log[H^+] < -\log 10^{-7} = 7$

* Genetic mutation: The basic source of genetic diversity is mutation, or changes in the chemical structure of genes. If genes mutates at a constant rate m and if other evolutionary forces are negligible, then the frequency F of the original gene after t generation is

$$F = F_0 (1-m)^t$$

where F_0 is the frequency at $t = 0$

(a) Solve for t using log

$$\log F = \log F_0 + \log (1-m)^t = \log F_0 + t \log (1-m)$$

$$\therefore \quad t = \frac{\log (F/F_0)}{\log (1-m)}$$

(b) If $m = 5 \times 10^{-5}$, after how many generations $F/F_0 = 1/2$? $t \cong 13,863$.