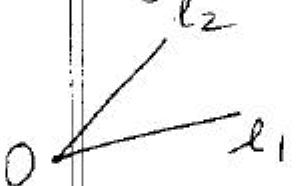
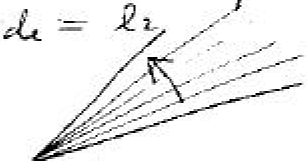
5.1 Angles



<u>angle</u> ≡ set of points (or region of plane) determined by 2 rays (or half-lines) l_1 and l_2 originating from the same point O

In <u>Trigonometry</u> we interpret an angle as rotation of rays

terminal side = l_2

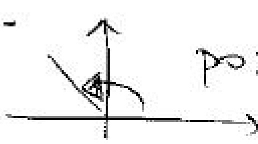


l_1 = initial side

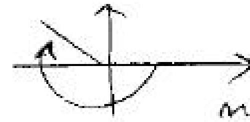
O vertex

Thus many angles might have the same initial and terminal side

Also, if l_1 is rotated counterclockwise we get a positive angle. If l_1 is rotated clockwise we get a negative angle.



positive



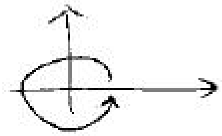
negative

As you could see in the picture above we usually introduce a rectangular coordinate system where

$O \equiv$ origin $\qquad l_1 \equiv x$-axis

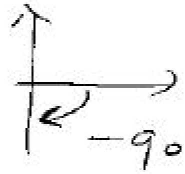
Measuring Angles: from high-school we are used to measure angles in degrees

e.g.

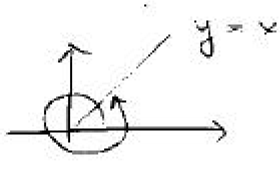


corresponds to $360°$

right angle $= 90°$

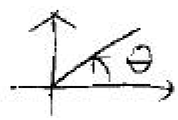


$-90°$



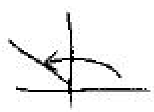
this is $45° + 360° = \underline{405°}$

Terminology:



$0° < \theta < 90°$

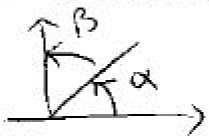
$\theta =$ acute angle



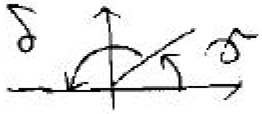
$90° < \beta < 180°$

$\beta =$ obtuse angle

two angles α, β such that $\alpha + \beta = 90°$ are said to be complementary angles



two angles γ, δ such that $\gamma + \delta = 180°$ are said to be supplementary angles



We also use smaller units to measure angles : $1° = 60'$ minutes and $1' = 60''$ seconds

Hence a typical angle is $47^\circ 15' 27''$

* Often though we see an angle given with decimal degrees:

57.2958° this can be rewritten as

$$57^\circ + (0.2958)^\circ = 57^\circ + 0.2958 \cdot \underbrace{1^\circ}_{60'}$$

$$= 57^\circ + 17.748' = 57^\circ + 17' + 0.748 \cdot 1' =$$

$$= 57^\circ + 17' + 0.748 \cdot 60'' = 57^\circ + 17' + 44.88''$$

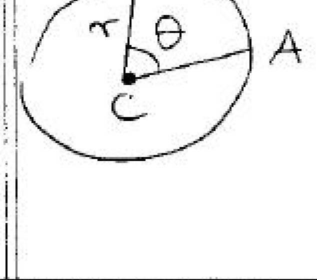
$$\cong 57^\circ 17' 45''$$

* Conversely an angle such as $19^\circ 47' 23''$

$$= 19^\circ + \left(\frac{47}{60}\right)^\circ + \left(\frac{23}{3600}\right)^\circ = 19^\circ + 0.7833^\circ + 0.0064^\circ$$

$$= (19.7897)^\circ$$

* In calculus we use instead the radian measure of an angle.
For that definition, let us consider



a central angle Θ (i.e. the vertex of Θ is the center of a circle) we say that Θ is subtended by the arc $\widehat{AP}$ OR $\widehat{AP}$ subtends the angle Θ.

If the length of $\widehat{AP}$ is r = radius then we say that θ has measure 1 radian.

Thus the radian measure of an angle

$$= \frac{\text{length of arc } \widehat{AP}}{\text{radius}}$$

Since the length of a circumference is $2\pi r$ then

$$360° \longleftrightarrow \frac{2\pi r}{r} = 2\pi \text{ radians}$$
$$\doteq 6.28 \text{ radians}$$

$$\frac{360°}{2\pi} = (57.2958)° \longleftrightarrow 1 \text{ radian}$$

$$1° \longleftrightarrow \frac{\pi}{180} \simeq 0.0175 \text{ radian}$$

* Ex:

$$30° \longleftrightarrow 30° \cdot \frac{\pi}{180°} = \frac{\pi}{6}$$

$$60° \longleftrightarrow 60° \cdot \frac{\pi}{180°} = \frac{\pi}{3}$$

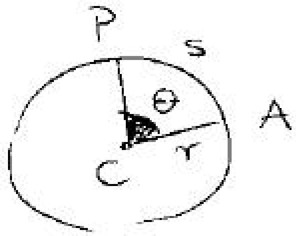
$$90° \longleftrightarrow 90° \cdot \frac{\pi}{180°} = \pi/2$$

$$\theta° \longleftrightarrow \theta° \cdot \frac{\pi}{180°} \text{ radian}$$

Conversely angle α measured in radian

$\longleftrightarrow \alpha \cdot \frac{180°}{\pi}$ in degrees.

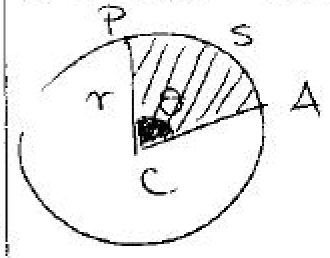
* Length of an arc



we saw that the radian measure of the angle subtended by the arc $\widehat{AP}$ is

$$\theta = \frac{s}{r} \quad \text{OR} \quad \boxed{s = r\theta}$$

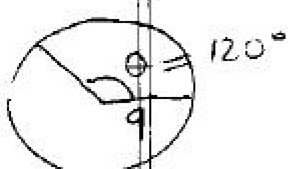
whereas the area of a circular sector is



$$A = \frac{1}{2}\theta r^2 = \frac{1}{2}\, r \cdot \underbrace{r\theta}_{s} = \frac{1}{2}\, r \cdot s$$

(like the area of a triangle!)

*** Ex:** Suppose that θ is a central angle of $120°$ of a circle with radius 9 cm. Find the length of the arc and the area of the sector

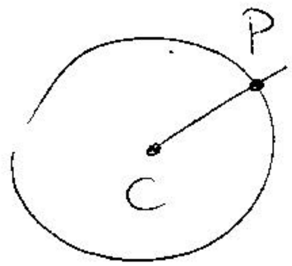


$$\theta = 120° \cdot \frac{\pi}{180°} = \frac{2}{3}\pi \text{ in radian}$$

$$\therefore\ s = r\theta = 9 \cdot \frac{2}{3}\pi = 6\pi \cong 18.84 \text{ cm}$$

$$\therefore\ A = \frac{1}{2}\theta r^2 = \frac{1}{2}\cdot\frac{2}{3}\pi \cdot 81 = 27\pi \text{ cm}^2 \cong 84.82 \text{ cm}^2$$

* Angular Speed of a wheel that is rotating at a constant rate is the angle generated in one unit of time by a line segment from the center of the wheel to a point P on the circumference.



The linear speed is the distance travelled by a point P on the circumference.

* Ex: a wheel of diameter 3 feet rotates at 1,600 rpm (revolution per minutes)

its angular speed is $= 1600 \cdot 2\pi =$

$= 3{,}200\pi$ radian per minute

its linear speed is $s = r \cdot \theta = \frac{3}{2} \cdot 3200\pi$ feet

$= 4800\pi$ feet

$\cong 15{,}079.64$ feet