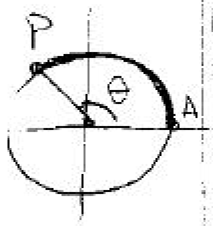
5.3 Trigonometric functions of Real numbers

The value of a trigonometric function at a real number t is its value at an angle whose radian measure is t (provided that the value exists)



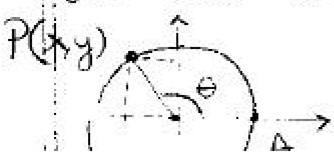
Actually, by considering a unit circle $(r=1)$ then the arc length of AP becomes $s = r\theta \rightsquigarrow s = \theta$. Thus, our real number $t (=s)$ may be regarded both as the radian measure of an angle θ or as the length of a circular arc AP

For $t > 2\pi$ we keep going around the circle (counterclockwise)

For $t < 0$ we keep going around the circle (clockwise)

As a consequence the Trigonometric function repeat their values on a regular basis. They are said to be period functions.

Hence for each $t \in \mathbb{R}$ there is a unique $P(x,y)$ on the unit circle, i.e. $x^2+y^2=1$, and



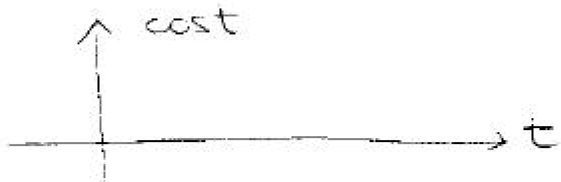
$t = \theta$ (as $r=1$)

$\sin t = y \quad \cos t = x$

and consequently $\tan t = \frac{\sin t}{\cos t} = \frac{y}{x} \quad (x \neq 0)$

$\cot t = \frac{x}{y} \quad (y \neq 0) \qquad \sec t = \frac{1}{x} \qquad \csc t = \frac{1}{y}$

Our next goal is to <u>graph</u> ~~the~~ <u>trigonometric functions</u> in the cartesian plane (since x has another meaning, we will replace the x-axis with a t-axis)



on the vertical axis we will record the values of $\cos t$.

Consider the point $P(x,y)$ or better $P(\cos t, \sin t)$ travelling around the unit circle for t in the interval $[0, 2\pi]$.

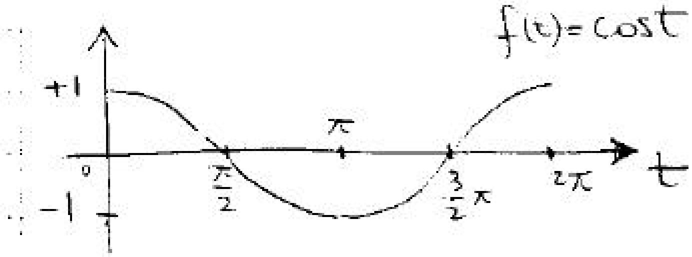
We have:

$\cos(0) = 1, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}$

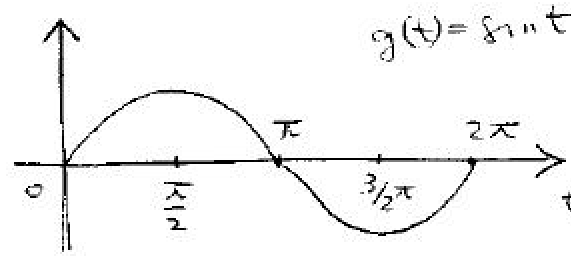
$\cos(\pi/2) = 0, \quad \cos(\underbrace{\tfrac{2}{3}\pi}_{\frac{\pi}{2}+\frac{\pi}{6}}) = -\frac{1}{2}$ why? $, \quad \cos(\underbrace{\tfrac{3}{4}\pi}_{\frac{\pi}{2}+\frac{\pi}{4}}) = -\frac{\sqrt{2}}{2}$ why?

$\cos(\underbrace{\tfrac{5}{6}\pi}_{\frac{\pi}{2}+\frac{\pi}{3}}) = -\sqrt{3}/2$ why? $, \quad \cos(\pi) = -1, \quad$ etc....

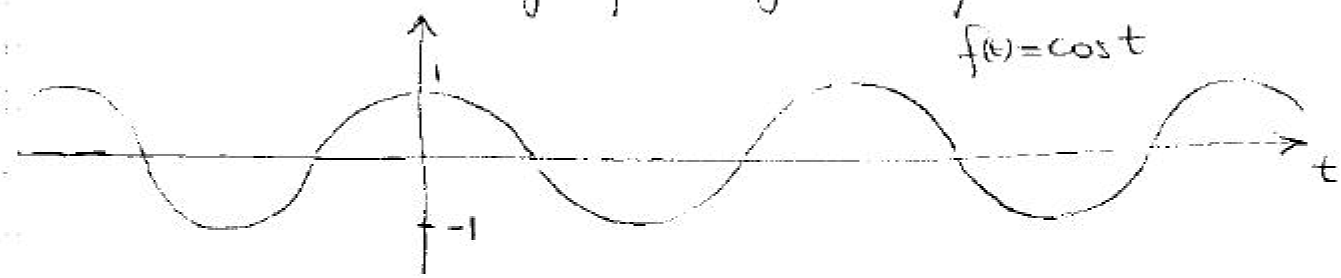
The graph looks like :

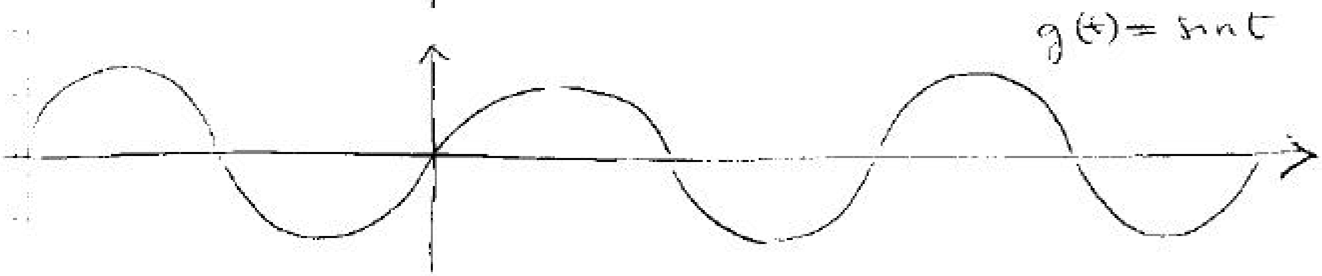


Similarly $\sin t$:



For values of t greater than 2π or smaller than 0 the graphs get repeated .





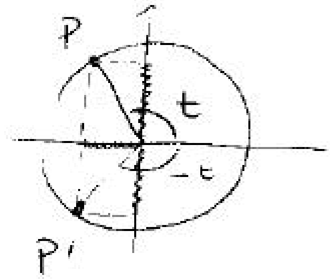
So, notice that

$$\cos(t + 2\pi k) = \cos t$$
$$\sin(t + 2\pi k) = \sin t \qquad \text{for } k \in \mathbb{Z}$$

i.e. they are periodic of period 2π

Also $\cos(-t) = \cos t$ (even)

$\sin(-t) = -\sin t$ (odd)



Thus $\cos t$ is an even function whereas $\sin t$ is an odd function. Thus

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\tan t \quad (\underline{\text{odd}})$$

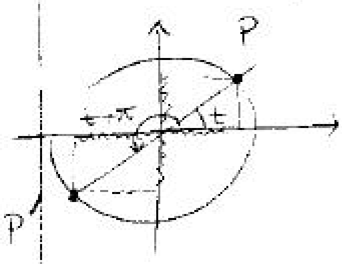
$$\cot(-t) = \frac{1}{\tan(-t)} = -\frac{1}{\tan t} = -\cot(t) \quad (\underline{\text{odd}})$$

finally $\sec t = \frac{1}{\cos t}$ is even and $\csc t$ is odd.

What about the other trigonometric functions?

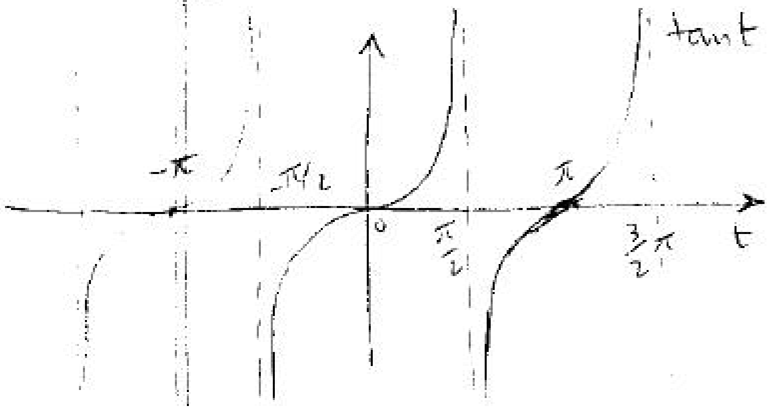
Look at $\tan t$, we first observe that its period is smaller: it is $\underline{\pi}$

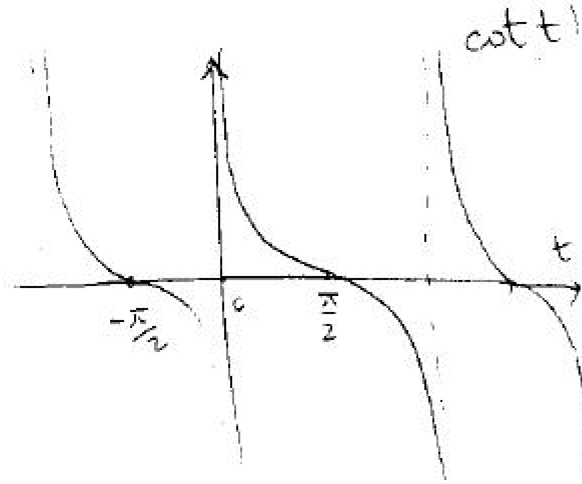
$$\tan(t+\pi) = \frac{\sin(t+\pi)}{\cos(t+\pi)} = \frac{-\sin t}{-\cos t} = \tan t$$

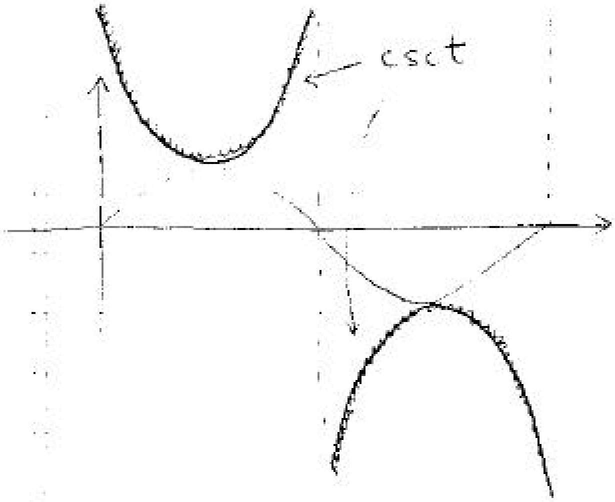


Similarly for $\cot(t)$.

It turns out that







this is the graph of

$\csc(t) = \frac{1}{\sin t}$

Notice that there are vertical asymptotes at $t = 0, \pi, 2\pi$, etc....

* Verify that $\sin(-t)\sec(-t) = -\tan t$

$$\sin(-t) \cdot \frac{1}{\cos(-t)} = -\sin(t) \cdot \frac{1}{\cos(t)} = -\frac{\sin t}{\cos t} = -\tan t \checkmark$$

* From the graphs of the trig. functions

if $t \to 0^+$ then $\sin t \to ?$: $\sin t \to 0$

if $t \to \pi^-$ then $\csc t \to +\infty$

if $t \to \pi^+$ then $\csc t \to -\infty$

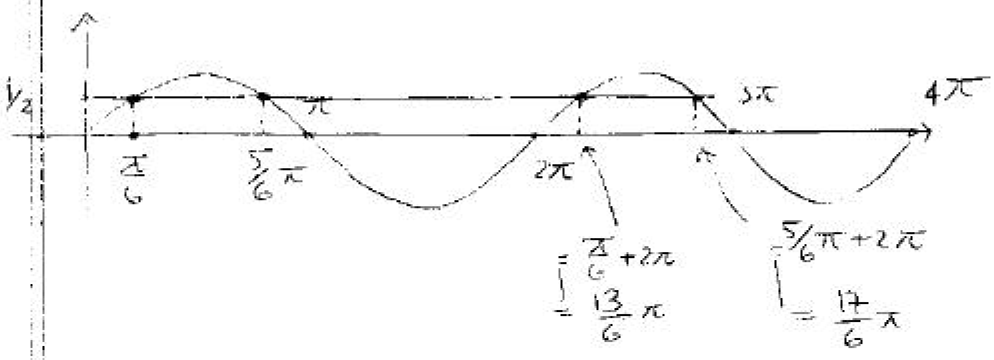
if $t \to \pi/2^+$ then $\tan t \to -\infty$

if $t \to \frac{\pi}{2}^-$ then $\tan t \to +\infty$

if $t \to \pi/4^+$ then $\tan t \to 1$

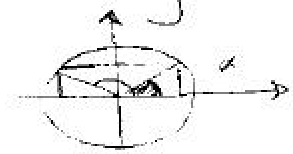
* Refer to the graph of $\sin t$ or $\cos t$ to find the exact values of t in $[0, 4\pi]$

such that $\sin t = 1/2$



Observe that for supplementary angles

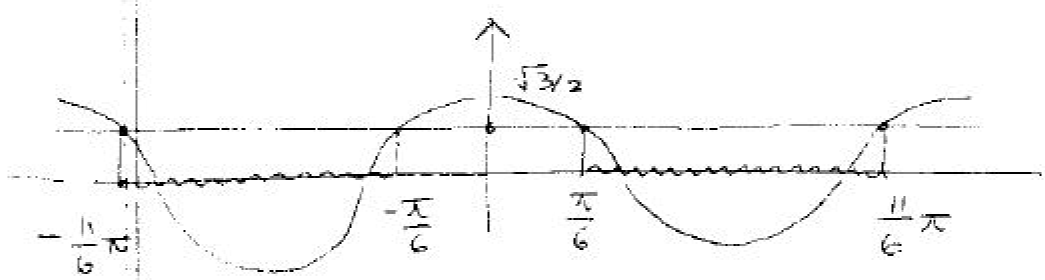
$$\sin \alpha = \sin(\pi - \alpha)$$



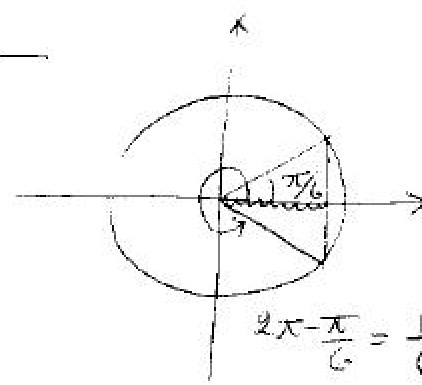
That's why $\sin(\pi/6) = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(\frac{5}{6}\pi\right) = \frac{1}{2}$

* Find all values of t in $[-2\pi, 2\pi]$ such that

$$\cos t \leq \sqrt{3}/2$$



because



$$2\pi - \frac{\pi}{6} = \frac{11}{6}\pi$$