6.3 The Addition and Subtraction Formulas

What if we want to compute $\cos(75°)$?

Notice $\cos(75°) = \cos(45° + 30°) = 0.2588 =$ according to your calculator. I claim $= \dfrac{\sqrt{6}-\sqrt{2}}{4}$. why?

In radians $\cos\left(\frac{5}{12}\pi\right) = \cos\left(\pi/4 + \pi/6\right)$.

Euler's formula says: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

where i is the imaginary number $i^2 = -1$.

Thus:

$$e^{i(u+v)} = e^{iu} \cdot e^{iv}$$

$$= (\cos u + i\sin u)(\cos v + i\sin v)$$

$$= (\cos u\cos v - \sin u\sin v) + i(\sin u\cos v + \cos u\sin v)$$

$$e^{i(u+v)} = \cos(u+v) + i\sin(u+v)$$

Thus:

$$\cos(u+v) = \cos u\cos v - \sin u\sin v$$

$$\sin(u+v) = \cos u\sin v + \sin u\cos v \qquad \text{Hence...}$$

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\cos(u)\sin v + \sin u\cos v}{\cos u\cos v - \sin u\sin v}$$

$$= \cdots = \frac{\tan u + \tan v}{1 - \tan u\tan v}$$

What about the trig. functions of "$u-v$"?
We use the fact that cosine is an <u>even</u> function whereas sine and tangent are <u>odd</u>

$$\cos(u-v) = \cos(u+(-v)) = \cos u \cdot \cos(-v) - \sin(u)\sin(-v)$$
$$\boxed{= \cos u \cos v + \sin u \sin v}$$

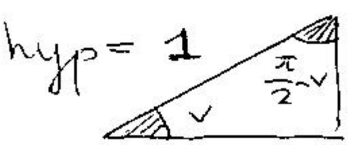
$$\sin(u-v) = \cos(u)\sin(-v) + \cos(-v)\sin(u)$$
$$\boxed{= \sin u \cos v - \cos u \sin v}$$

$$\tan(u-v) = \frac{\tan(u) + \tan(-v)}{1-\tan(u)\tan(-v)} = \frac{\tan u - \tan v}{1+\tan u \tan v}$$

* <u>Examples</u>.

$$\cos(\pi/2 - v) = \underbrace{\cos\frac{\pi}{2}}_{0}\cos v + \underbrace{\sin\frac{\pi}{2}}_{=1}\sin v = \sin v$$

or otherwise consider the right triangle

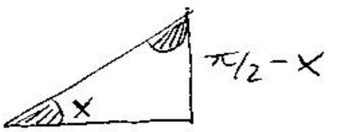


$\leftarrow$ this is $\cos(\pi/2 - v)$
<u>but</u> for the angle v that is $\sin v$

* Similarly $\sin(\pi/2 - v) = \cos v$

* Show that $\tan(x - \pi/2) = -\cot x$

In this case we cannot use the subtraction formula since $\tan(\pi/2)$ is not defined. However,



$$\tan(x - \pi/2) \overset{\text{odd function}}{=} -\tan(\pi/2 - x)$$

$$= -\cot x$$ because of the right Triangle on the left

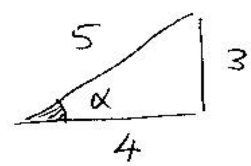
* If α and β are acute angles such that $\cos\alpha = 4/5$ and $\tan\beta = 8/15$ find

(a) $\sin(\alpha+\beta)$

(b) $\cos(\alpha+\beta)$

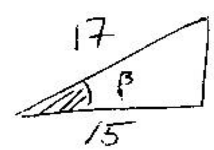
(c) the quadrant containing $\alpha+\beta$

Ans: $\cos\alpha = 4/5$ leads to



so that $\sin\alpha = 3/5$

Similarly, $\tan\beta = 8/15$ leads to



thus $\cos\beta = 15/17$ and $\sin\beta = 8/17$

$\therefore \sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta = \frac{4}{5}\frac{8}{17} + \frac{3}{5}\frac{15}{17}$

$\quad = 77/85$

$\therefore \cos(\alpha+\beta) = \dots = 36/85$ $\Big\}$ $\therefore \alpha+\beta$ is in the first quadrant

* Verify that $\sin\left(\theta + \pi/4\right) = \sqrt{2}/2\,(\sin\theta + \cos\theta)$

Ans :

$$\sin\left(\theta + \frac{\pi}{4}\right) = \cos\theta\,\sin\frac{\pi}{4} + \sin\theta\,\cos\pi/4$$

$$= \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta$$

$$= \frac{\sqrt{2}}{2}\left[\sin\theta + \cos\theta\right] \quad \checkmark$$

* Notice that $\cos(2x) = \cos^2 x - \sin^2 x$. Thus if we need to find the solutions of the equation

$$\cos 2x + 3\cos x + 2 = 0 \quad \text{in } [0, 2\pi)$$

We proceed as follows.

Ans : $(\cos^2 x - \sin^2 x) + 3\cos x + 2 = 0$

get rid of $\sin^2 x = 1 - \cos^2 x$

$$2\cos^2 x - 1 + 3\cos x + 2 = 0$$

$$2\cos^2 x + 3\cos x + 1 = 0$$

$$(2\cos x + 1)(\cos x + 1) = 0$$

$$\iff \quad \cos x = -1/2 \qquad \cos x = -1$$

$$x = \frac{2}{3}\pi,\ 4/3\,\pi \qquad x = \pi$$