6.4 Multiple-Angle Formulas

Let us consider the addition formulas for $\cos(u+v)$ and $\sin(u+v)$

$$\begin{cases} \cos(u+v) = \cos u \cos v - \sin u \sin v \\ \sin(u+v) = \cos u \sin v + \sin u \cos v \end{cases}$$

and set $u = v$. We get that $u+v = 2u$ so that

$$* \quad \underline{\cos(2u)} = \underline{\cos^2 u - \sin^2 u}$$

$$\overset{!}{=} (1-\sin^2 u) - \sin^2 u = \underline{1 - 2\sin^2 u}$$

$$\overset{!}{=} \cos^2 u - (1-\cos^2 u) = \underline{2\cos^2 u - 1}$$

$$* \quad \underline{\sin(2u)} = \underline{2 \sin u \cos u}$$

We can also look at $\tan(2u)$. The formula in this case becomes

$$\underline{\tan(2u)} = \underline{\frac{2\tan u}{1-\tan^2 u}}$$

These are called the <u>double</u> angle <u>formulas</u>

From $\cos(2u)$ we can also deduce the <u>half-angle identities</u>:

In fact $\cos(2u) = 1 - 2\sin^2 u$ or $= 2\cos^2 u - 1$

In one case we get

$$\cos(2u) = 1 - 2\sin^2 u \iff \sin^2 u = \frac{1 - \cos(2u)}{2}$$

set $2u = \theta$ or $\theta/2 = u$

$$\iff \sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}$$

$$\cos(2u) = 2\cos^2 u - 1 \iff \cos^2 u = \frac{1 + \cos(2u)}{2}$$

$$\iff \cos^2\frac{\theta}{2} = \frac{1 + \cos\theta}{2}$$

Thus:

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}}$$

where $\pm$ is decided by the quadrant in which $\theta/2$ is located.

What about $\tan\frac{\theta}{2} = ?$

$$\tan^2\frac{\theta}{2} = \frac{\sin^2\theta/2}{\cos^2\theta/2} = \frac{\frac{1-\cos\theta}{2}}{\frac{1+\cos\theta}{2}} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$= \frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$\boxed{\tan\theta/2 = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \pm\frac{1-\cos\theta}{\sin\theta}}$$

* We have already observed that

$\cos(75^\circ) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ by using the

addition formulas with $75^\circ = 45^\circ + 30^\circ$

Observe that $75^\circ = \dfrac{150^\circ}{2}$. Thus

$$\cos(75^\circ) = \cos\left(\frac{150^\circ}{2}\right) = \underset{\substack{\uparrow \\ 75^\circ \text{ is in the first quadrant}}}{+}\sqrt{\frac{1+\cos 150^\circ}{2}}$$

$$= \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

<u>Well</u>, think why $\dfrac{\sqrt{6}-\sqrt{2}}{4} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$!!

$\cong 0.2588$

* <u>Fact</u> one can prove the following radical formulas:

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

<u>eg.</u> $\sqrt{2-\sqrt{3}} = \sqrt{\dfrac{2+\sqrt{4-3}}{2}} - \sqrt{\dfrac{2-\sqrt{4-3}}{2}}$

$$= \sqrt{\frac{2+1}{2}} - \sqrt{\frac{2-1}{2}} = \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}} = \frac{\sqrt{6}-\sqrt{2}}{2}$$

* We can use the half-angle formulas to reduce powers of $\cos t$.

$$\underline{\cos^4 t} = (\cos^2 t)^2 = \left(\frac{1+\cos 2t}{2}\right)^2 =$$

$$= \frac{1}{4} + \frac{1}{2}\cos(2t) + \frac{1}{4}\cos^2(2t)$$

$$= \frac{1}{4} + \frac{1}{2}\cos(2t) + \frac{1}{4}\left(\frac{1+\cos(4t)}{2}\right)$$

$$= \cdots = \underline{\frac{3}{8} + \frac{1}{2}\cos(2t) + \frac{1}{8}\cos(4t)}$$

* What about $\cos(3u) = ?$

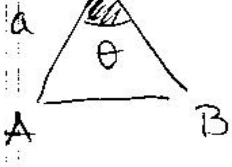
$$\underline{\cos(3u)} = \cos(2u+u) = \cos(2u)\cos(u) - \sin(2u)\sin(u)$$

$$= (\cos^2 u - \sin^2 u)\cos u - (2\sin u \cos u)\sin(u)$$

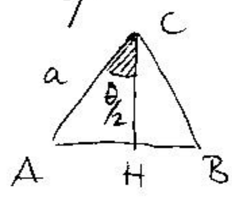
$$= (2\cos^2 u - 1)\cos u - 2\cos u(1-\cos^2 u)$$

$$= \underline{4\cos^3 u - 3\cos u}$$

* Express the area of the isosceles triangle in terms of $a = \overline{AC} = \overline{CB}$ and θ.



Ans:



$\overline{AH} = a\sin\theta/2$

$\overline{CH} = a\cos\theta/2$

observe that $\text{area} = \frac{1}{2}\overline{AB}\cdot\overline{CH} = \frac{1}{2}(2\overline{AH})\cdot\overline{CH}$

$$= \overline{AH}\cdot\overline{CH} = \left[a\left(\sin\frac{\theta}{2}\right)\right]\left[a\cos\frac{\theta}{2}\right] = \boxed{a^2\,\frac{1}{2}\sin\theta}$$