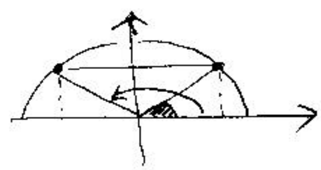
We skip sections 6.5 and 6.6

but we cover sections 7.1 and 7.2

7.1 The Law of Sines

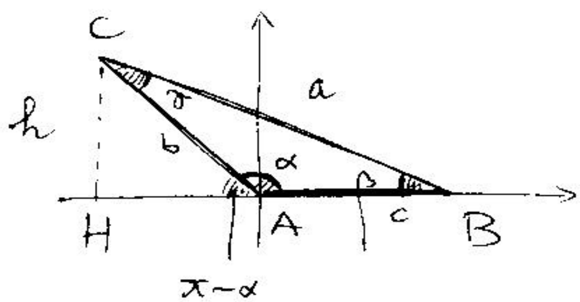
7.2 The Law of Cosines

Recall that :



$$\sin(\alpha) = \sin(\pi - \alpha) \overset{!}{=} \sin(180 - \alpha)$$

Now, consider the triangle



and observe that

$$\sin\alpha = \sin(\pi - \alpha) \overset{!}{=} h/b \quad \text{from } \widehat{ACH}$$

whereas $\sin\beta = \dfrac{h}{a}$ from $\widehat{BCH}$

$\therefore \quad a \sin\beta = h = b\sin\alpha$ Which can be rewritten as

$$\boxed{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}}$$

Similarly, $\boxed{\dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}}$ $\boxed{\dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}}$

Those 3 formulas are dubbed "Law of Sines".

We can also write them

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

* Solve a triangle given that

$a = 12.4$, $b = 8.7$ and $\beta = 36.7°$

Ans: From $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$ we obtain

$$\sin\alpha = \frac{a}{b}\sin\beta = \frac{12.4}{8.7}\sin(36.7°) \cong 0.8518$$

Note that in $[0,\pi]$ there are 2 angles with the same sine : namely α, and $\pi-\alpha$ (comment at the beginning of class).

the calculator gives $\alpha_1 = \sin^{-1}(0.8518) \doteq 58.4°$

but we also need to consider $\alpha_2 = 180 - 58.4° \doteq 121.6°$

if $\alpha_1 = 58.4°$ then

$\gamma_1 = 180° - \alpha_1 - \beta = \doteq 84.9$

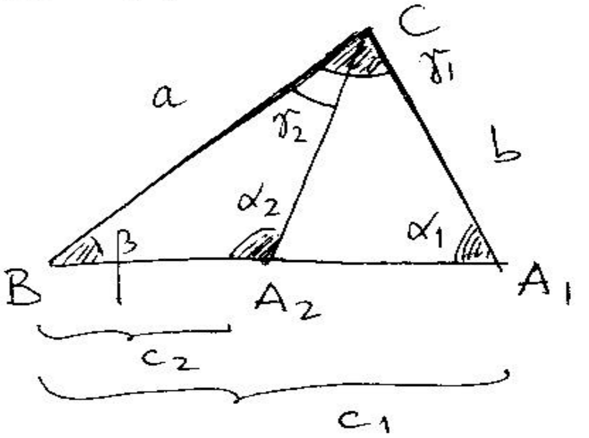
$c_1 = a\frac{\sin\gamma_1}{\sin\alpha_1} \cong 14.5$

if $\alpha_2 = 121.6°$ then

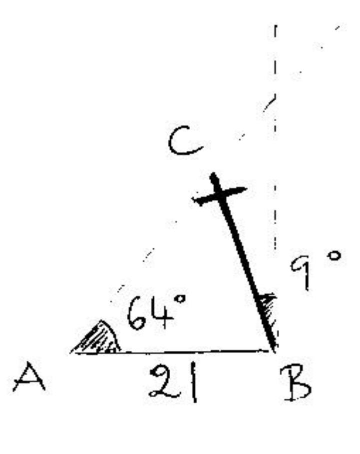
$\gamma_2 = 180° - \alpha_2 - \beta = 21.7°$

$c_2 = a\frac{\sin\gamma_2}{\sin\alpha_2} \cong 5.4$

The picture of the 2 possible triangles is



When the angle of the sun in 64° a telephone pole that is tilted at an angle of 9° directly away from the sun casts a shadow of 21 feet long on level ground. Approximate the length of the pole.



Observe that $\hat{A} = 64°$

$\hat{B} = 90° - 9° = 81°$

$\hat{C} = 180° - 64° - 81° = 35°$

Thus using the law of sines

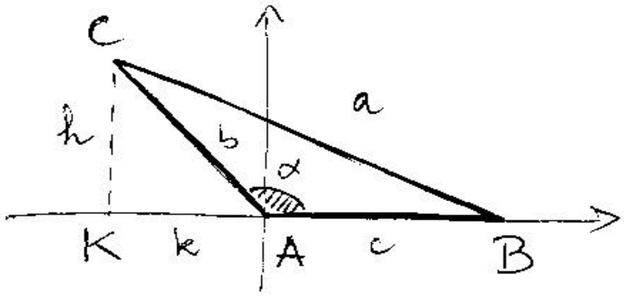
$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightsquigarrow \frac{\sin 64°}{\text{length of pole}} = \frac{\sin 35°}{21}$$

$\therefore$ length of pole $= 21\,\dfrac{\sin 64°}{\sin 35°} \cong 33$ feet

We <u>cannot</u> use the law of sines when we are given :

- 2 sides and the angle between them
- 3 sides

In these cases we use the <u>law of cosines</u>



in the cartesian plane assume that

$A(0,0) \quad B(c,0) \quad C(h,k) \quad K(k,0)$

so $c > 0$, $h > 0$, $\underline{k < 0}$

Notice that $\cos\alpha = \frac{k}{b}$ (negative!)

$$\sin\alpha = h/b$$

$\therefore \quad k = b\cos\alpha \quad h = b\sin\alpha$

<u>Pytagoras</u> <u>theorem</u> applied to $\widehat{BCK}$ says:

$$\underbrace{dist(B,C)^2}_{a^2} = dist(B,K)^2 + dist(C,K)^2$$

$$\stackrel{!}{=} (c-k)^2 + h^2$$

$$\stackrel{!}{=} (c - b\cos\alpha)^2 + (b\sin\alpha)^2$$

Thus: $a^2 = (c - b\cos\alpha)^2 + (b\sin\alpha)^2$

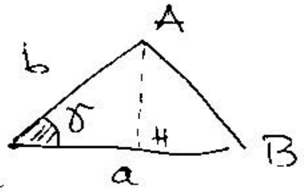
$$= c^2 - 2bc\cos\alpha + \underbrace{b^2\cos^2\alpha + b^2\sin^2\alpha}_{= b^2(\cos^2\alpha + \sin^2\alpha) = b^2 \cdot 1}$$

$\therefore$ $\boxed{a^2 = b^2 + c^2 - 2bc\cos\alpha}$

Similarly we obtain that

$$\boxed{\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}}$$

* When we know 2 sides and the angle in between we also have that the area of the triangle is



$$\boxed{\text{Area} = \frac{1}{2}ab\sin\gamma}$$

when $\overline{AH} = b\sin\gamma$

* when we know the 3 sides a, b, c of the triangle, we can use the law of cosines to prove that the area of the triangle is

$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a+b+c)$ = half perimeter

This is known as <u>Heron's Formula</u>

* <u>Ex</u>: If a triangle ABC has sides $a = 90$, $b = 70$ and $c = 40$ approximate the angles α, β and γ to the nearest degree.

<u>Ans</u>: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

gives $\cos\alpha = \dfrac{b^2+c^2-a^2}{2bc} = \dots = -2/7$

thus $\alpha = \cos^{-1}(-2/7) \cong 107°$

Similarly $\cos\beta = \dfrac{a^2+c^2-b^2}{2ac} = \dots = 2/3$

thus $\beta = \cos^{-1}(2/3) \cong 48°$

hence $\gamma = 180° - 107° - 48° = 25°$.