1.4 Equations

We'll consider
- linear equations : $ax+b=0$
- quadratic equations : $ax^2+bx+c=0$

Let's work out several examples :

* $5x-4=2(x-2)$ we want to bring it to the standard form $ax+b=0$

$5x-4-2(x-2)=0$

$5x-4-2x+4=0$ $3x=0 \rightsquigarrow x=0$

* $\dfrac{5x+2}{10x-3}=\dfrac{x-8}{2x+3}$ we cross multiply

$(2x+3)(5x+2)=(x-8)(10x-3)$ [Note $x \neq \frac{3}{10}, -\frac{3}{2}$]

it looks quadratic ... but....

$\cancel{10x^2}+4x+15x+6=\cancel{10x^2}-3x-80x+24$

$19x+6+83x-24=0$ $104x-18=0$

$\therefore\ x=\dfrac{18}{104}=\dfrac{3}{17}$

Let's come to quadratic equations :
$ax^2+bx+c=0$

* Solve $3x^2+x-10=0$. It is a matter of factoring it (if possible).

$\longleftrightarrow\ (3x-5)(x+2)=0 \longrightarrow x=\frac{5}{3}$ or $x=-2$

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What if we can't factor it so easily?

We need the quadratic formula. An example first, though!

* Solve $(x+4)^2 = 31$

$$\rightsquigarrow \sqrt{(x+4)^2} = \sqrt{31} \quad \rightsquigarrow \quad x+4 = \pm\sqrt{31}$$

$$\sqrt{(x+4)^2} = |x+4|$$

so $x = -4 \pm \sqrt{31}$ (2 solutions!)

In general from $ax^2+bx+c=0$ we complete the squares (like above) and then we get the formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$\Delta = b^2-4ac$ is the <u>discriminant</u>

* $3x^2+x-10 = 0 = (3x-5)(x+2)$ Revisited !!

$$\hookrightarrow x_{1,2} = \frac{-1 \pm \sqrt{1^2-4\cdot 3\cdot(-10)}}{2\cdot 3} = \frac{-1 \pm \sqrt{121}}{6} = \begin{cases} \frac{-1-11}{6} = -2 \\ \frac{-1+11}{6} = \frac{5}{3} \end{cases}$$

$\therefore$ as factorization $(x-5/3)(x+2) = 0$

clear denominators $(3x-5)(x+2)=0$.

* Factor $15x^2+34x-16=0$

$$x_{1,2} = \frac{-34 \pm \sqrt{34^2-4\cdot 15(-16)}}{2\cdot 15} = \frac{-34 \pm \sqrt{1156+960}}{30}$$

$$= \frac{-34 \pm \sqrt{2116}}{30} = \frac{-34 \pm 46}{30} \begin{cases} \frac{12}{30} = \frac{2}{5} \\ -\frac{80}{30} = -\frac{8}{3} \end{cases}$$

As factorization $\left(x + \frac{8}{3}\right)\left(x - \frac{2}{5}\right) = 0$

or $(3x + 8)(5x - 2) = 0$

* Solve $2|5x+2| - 1 = 5 \longrightarrow 2|5x+2| = 6$

$|5x+2| = 3 \longrightarrow 5x + 2 = \pm 3 \begin{cases} 5x+2 = 3 \\ 5x+2 = -3 \end{cases}$

$\begin{cases} x = +\frac{1}{5} \\ x = -1 \end{cases}$

* $4x^4 + 10x^3 = 6x^2 + 15x$

$x\left[\underbrace{4x^3 + 10x^2} - 6x - 15\right] = 0$ group

$x\left[2x(2x^2 - 3) + 5(2x^2 - 3)\right] = 0$

$\therefore x \cdot (2x+5)(2x^2-3) = 0$

$\longrightarrow x = 0$ or $x = -\frac{5}{2}$ or $2x^2 - 3 = 0 \longleftrightarrow$

$x^2 = \frac{3}{2} \longrightarrow x = \pm \frac{\sqrt{3}}{\sqrt{2}}$

or after rationalizing $x = \pm \frac{\sqrt{6}}{2}$.

* $3y^4 - 5y^2 + 1 = 0$ Think of it as

$3(y^2)^2 - 5(y^2) + 1 = 0$ Set $x = y^2$

$3x^2 - 5x + 1 = 0$

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$$x_{1,2} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 3\cdot 1}}{2\cdot 3} = \frac{5 \pm \sqrt{25-12}}{6} = \frac{5 \pm \sqrt{13}}{6}$$

both positive

But $x_{1,2} = y^2$ So there are 4 solutions

$$y_{1,2} = \sqrt{\frac{5 \pm \sqrt{13}}{6}} = \frac{1}{6}\sqrt{30 \pm 6\sqrt{13}}$$

$$y_{3,4} = -\sqrt{\frac{5 \pm \sqrt{13}}{6}} = -\frac{1}{6}\sqrt{30 \pm 6\sqrt{13}}$$

* $2y^{1/3} - 3y^{1/6} + 1 = 0$ Same as before

$2\left(y^{1/6}\right)^2 - 3y^{1/6} + 1 = 0$ set $y^{1/6} = x$

$2x^2 - 3x + 1 = 0$ $(2x-1)(x-1) = 0$

or $2x - 1 = 0$ or $x - 1 = 0$

$\therefore x = \frac{1}{2}$ or $x = 1$ but $y^{1/6} = x$

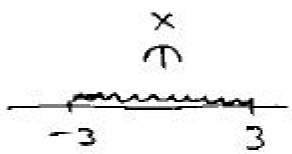
$\iff y = x^6$

$\therefore y = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$ or $y = 1^6 = 1$

* $\sqrt{3-x} - x = 3 \iff \sqrt{3-x} = 3 + x$

Notice that the solution x must be s/t

$3 - x \geq 0$ or $x \leq 3$



on the other hand $3 + x \geq 0$

$\therefore x \geq -3$

Now, $\sqrt{3-x} = 3+x$ square it!

$3-x = (3+x)^2 \qquad 3-x = 9+6x+x^2$

or $x^2+7x+6=0 \qquad (x+6)(x+1)=0$

$\longrightarrow \cancel{x=-6}$ or $x=-1$

* Consider the equation

$$CD + C = PC + N$$

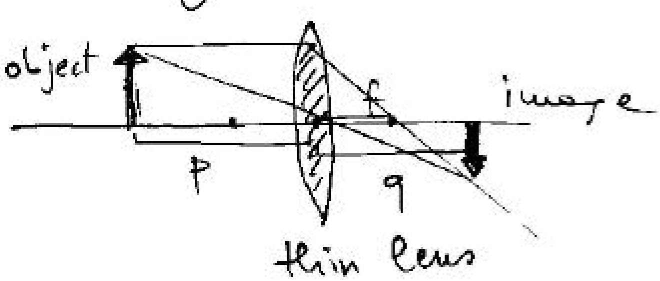
Solve it for C.

$$CD + C - PC = N$$

$$C(D+1-P) = N \qquad \text{so} \qquad C = \frac{N}{D+1-P}$$

* Solve for q in the lens equation

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$



$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{p-f}{fp}$$

$$\therefore \quad q = \frac{fp}{p-f}$$

* A city government has approved the construction of a $50 million sports arena. Up to $30 million will be raised by

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selling bonds that pay simple interest at a rate of 12% annually. The remaining (up to $40 million) will be obtained by borrowing money from an insurance company at a simple rate of 10%.

Determine whether the area can be financed so that the annual interest is $5.2 million.

Answer:

$x =$ money in bonds (million) ≤ 30

$50 - x =$ money borrowed (million) $\geqslant 40$

$$\underbrace{0.12x} + \underbrace{0.1(50-x)} = \text{interest} = 5.2$$

$$0.12x + 5 - 0.1x = 5.2$$

$$0.02x = 0.2 \qquad x = \frac{0.2}{0.02} = \boxed{10}$$