1.5 Complex Numbers

We need complex numbers to solve equations such as: $\boxed{x^2 + 9 = 0} \longrightarrow x^2 = -9$

We introduce the imaginary number $i = \sqrt{-1}$

or $i^2 = -1 \iff i = \sqrt{-1}$

Thus $x^2 = -9 \longleftrightarrow x^2 = 9i^2 \longrightarrow \boxed{x = \pm 3i}$

A complex number in general is of the form $a + ib$ with $a, b \in \mathbb{R}$. We denote the complex numbers by $\mathbb{C}$.

We can add, subtract, multiply divide complex numbers.

* $(-3 + 8i) - (2 + 3i) = (-3-2) + i(8-3)$
$= -5 + 5i$

* $(8 + 2i)(7 - 3i) = 56 - 24i + 14i - 6i^2$

$= (56 + 6) + i(-24 + 14) = 62 - 10i$

* $i(2 - 7i)^2 = i\left[4 - 28i + 49i^2\right] =$

$= i\left[-45 - 28i\right] = -28i^2 - 45i = 28 - 45i$

22)

* $i^{73} = ?$ Notice that $i^2 = -1$

so $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$ So :

$i^{73} = i^{18 \cdot 4 + 1} = \left(i^4\right)^{18} \cdot i = (1)^{18} \cdot i = \underline{i}$

$73 = 18 \cdot 4 + 1$

* Complex conjugate : if $z = a + ib$ then the complex conjugate of z is

$\bar{z} = a - ib$

Why do we need it ? Well....

* Simplify $\dfrac{2+9i}{-3-i}$

we need to get rid of $-3-i$ at the denominator

$$\frac{2+9i}{-3-i} \cdot \frac{-3+i}{-3+i} = \frac{(2+9i)(-3+i)}{(-3-i)(-3+i)} =$$

$$= \frac{-6+2i-27i+9i^2}{(-3)^2-(i)^2} = \frac{-15-25i}{9+1} =$$

$$= \frac{-15}{10} - \frac{25}{10}i = -\frac{3}{2} - \frac{5}{2}i$$

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* $$\frac{5-\sqrt{-121}}{1+\sqrt{-25}} = \frac{5-11i}{1+5i} \cdot \frac{1-5i}{1-5i} =$$

$$= \frac{(5-11i)(1-5i)}{1^2-(5i)^2} = \frac{5-25i-11i+55i^2}{1+25}$$

$$= \frac{-50-36i}{26} = -\frac{25}{13} - \frac{18}{13}i$$

* Find the values of x and y such that

$$(x-y)+3i = 7+yi$$

Rewrite as :

$$(x-7-y)+i(3-y) = 0$$

$$\Rightarrow \quad x-7-y=0 \quad \text{and} \quad 3-y=0$$

so $\underline{y=3}$ and $\underline{x} = +7+y = 7+3 = \underline{10}$

* Solve the equation $x^2+8x+17=0$

$$x_{1,2} = \frac{-8 \pm \sqrt{64-4\cdot 17}}{2} = \frac{-8 \pm \sqrt{64-68}}{2} =$$

$$= \frac{-8 \pm \sqrt{-4}}{2} = \frac{-8 \pm 2i}{2} = \boxed{-4 \pm i}$$

24)

* $x^4 = 81 \rightsquigarrow x^4 - 81 = 0$

$(x^2)^2 - 9^2 = 0 \qquad (x^2+9)(x^2-9) = 0$

$\rightsquigarrow x^2 - 9 = 0 \longrightarrow \underline{x = \pm 3}$

$x^2 + 9 = 0 \longrightarrow x^2 = -9 \longrightarrow \underline{x = \pm 3i}$

* $8x^3 - 12x^2 + 2x - 3 = 0$

$4x^2(2x-3) + (2x-3) = 0$

$(2x-3)\left[4x^2+1\right] = 0 \rightsquigarrow 2x-3=0$ OR $4x^2+1=0$

$\rightsquigarrow \underline{x = 3/2}$ or $x^2 = -1/4 \rightsquigarrow \underline{x = \pm \frac{1}{2} i}$

* $x^3 - 27 = 0$ [recall $a^3 - b^3 = (a-b)(a^2+ab+b^2)$]

$x^3 - 27 = (x-3)(x^2+3x+9) = 0$

$\therefore x = 3$ OR $x^2 + 3x + 9 = 0 \qquad x_{1,2} = \dfrac{-3 \pm \sqrt{9-36}}{2}$

$= \dfrac{-3 \pm \sqrt{-27}}{2} = \dfrac{-3 \pm 3\sqrt{-3}}{2} = -\frac{3}{2} \pm \left(\frac{3}{2}\sqrt{3}\right) i$

* Verify $\overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad z, w \in \mathbb{C}$