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2.7 Operations on functions

Given 2 functions f and g we can define new functions

terminology $\longleftrightarrow$ function value

sum $f+g$ $\leftrightsquigarrow$ $(f+g)(x) = f(x)+g(x)$

difference $f-g$ $\leftrightsquigarrow$ $(f-g)(x) = f(x)-g(x)$

product fg $\leftrightsquigarrow$ $(fg)(x) = f(x)g(x)$

quotient f/g $\leftrightsquigarrow$ $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided $g(x) \neq 0$

The domain of $f+g$, $f-g$ and fg is the intersection of the domains of f and g. For the quotient f/g we have also to make sure that $g(x) \neq 0$.

* Example: given $f(x) = x^2+x$ and $g(x) = x^2-3$

then: $(f+g)(x) = f(x)+g(x) = (x^2+x)+(x^2-3)$

$$= 2x^2+x-3$$

$$(f-g)(x) = (x^2+x)-(x^2-3) = x+3$$

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$$(fg)(x) = f(x)g(x) = (x^2+x)(x^2-3)$$
$$= x^4 - 3x^2 + x^3 - 3x$$
$$= x^4 + x^3 - 3x^2 - 3x$$

In all these cases the domain of $f \pm g$, fg is still $\mathbb{R}$.

Finally, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2+x}{x^2-3}$

and the domain is $x \neq \pm\sqrt{3}$.

* Same thing for $f(x) = \sqrt{3-2x}$ and $g(x) = \sqrt{x+4}$.

$$f(x) \pm g(x) = \sqrt{3-2x} \pm \sqrt{x+4}$$

$$f(x)g(x) = \sqrt{x+4}\sqrt{3-2x}$$

the domain is the intersections of the domains of $f(x)$ and $g(x)$:

$g = \sqrt{x+4}$ has domain $x+4 \geq 0 \iff x \geq -4$

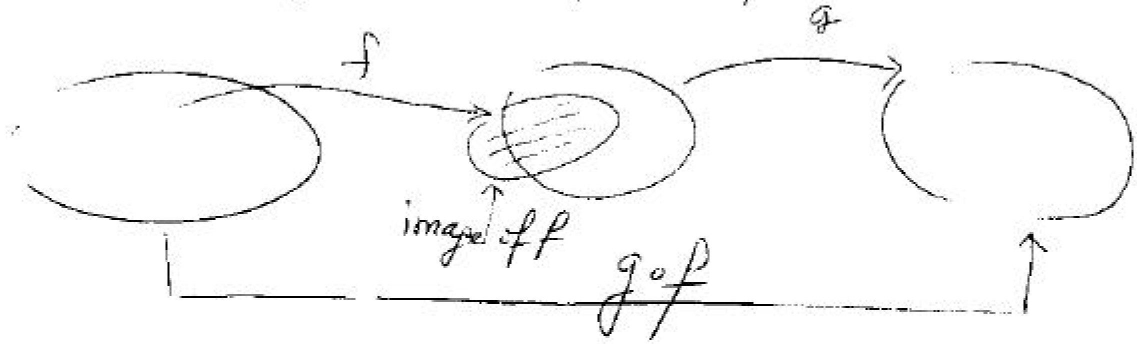
$f = \sqrt{3-2x}$ has domain $3-2x \geq 0 \iff x \leq \frac{3}{2}$

the intersection is $-4 \leq x \leq \frac{3}{2}$

69) the quotient $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{3-2x}{\sqrt{x+4}}}$

has domain $-4 < x \leq 3/2$

* Another operation on function is the composition of 2 functions.



$g \circ f$ is the function defined by

$$(g\circ f)(x) = g(f(x))$$

notice that the domain of $g \circ f$ consists of all x in the domain of f such that $f(x)$ is in the domain of g.

* Example

Let $f(x) = 3x^2$ $\qquad g(x) = x - 1$

Then $(f\circ g)(x) = f(g(x)) = f(x-1) =$

$$= 3(x-1)^2$$

$$(g \circ f)(x) = g(f(x)) = g(3x^2)$$
$$= 3x^2 - 1$$

$$(f \circ f)(x) = f(f(x)) = f(3x^2) = 3(3x)^2$$
$$= 27x^4$$

$$(g \circ g)(x) = x - 2$$

* Example

Same as above for $f(x) = |x|$ and $g(x) = -7$

$$(f \circ g)(x) = f(g(x)) = f(-7) = |-7| = 7$$
$$(g \circ f)(x) = g(f(x)) = g(|x|) = -7$$

Notice that we always get (in general)

$$f \circ g \neq g \circ f$$

* Example Find $f \circ g$ and $g \circ f$

if $f(x) = \sqrt{x - 15}$ and $g(x) = x^2 + 2x$

$$(f \circ g)(x) = f(g(x)) = \sqrt{(x^2 + 2x) - 15}$$

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What's the domain?

we need $x^2+2x-15 \geq 0$

$ \overset{\parallel}{(x+5)(x-3)}$

$\therefore$ domain $(-\infty,-5] \cup [3,+\infty)$

Instead, $(g\circ f)(x) = g(f(x)) = g(\sqrt{x-15})$

$= (\sqrt{x-15})^2 + 2(\sqrt{x-15})$

$= (x-15) + 2\sqrt{x-15}$ and so

the domain is $x \geq 15$

or $[15,+\infty)$

* Same for the functions

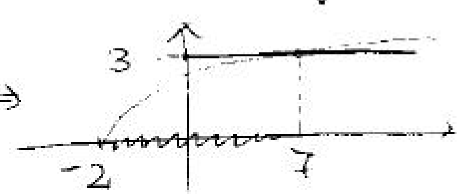
$f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x+2}$

we get $(f\circ g)(x) = f(g(x)) = \sqrt{3-g(x)}$

$= \sqrt{3-\sqrt{x+2}}$

For the domain we need $3-\sqrt{x+2} \geq 0$

or $\sqrt{x+2} \leq 3 \iff$



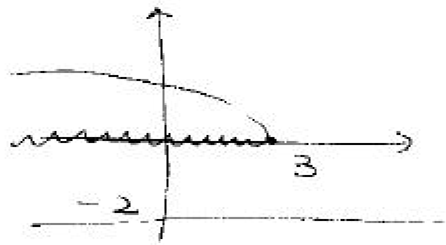
$\therefore$ domain $[-2,7]$

On the other hand

$$(g \circ f)(x) = g(f(x)) = \sqrt{f(x)+2} = \sqrt{\sqrt{3-x}+2}$$

we want $\sqrt{3-x}+2 \geq 0$ or

$$\sqrt{3-x} \geq -2$$



$\therefore$ domain $(-\infty, 3]$

* Finally Find a composite function form for

$$y = (x^2+3x)^{1/3}$$

$$x \xrightarrow{f} x^2+3x \xrightarrow{g} (x^2+3x)^{1/3}$$

$\therefore$ $g(x) = (x)^{1/3}$ $\qquad f(x) = x^2+3x$

so $y = (g \circ f)(x)$

* Same for $y = \dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}}$

$$x \xmapsto{f} \sqrt[3]{x} \xmapsto{g} \frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}$$

$y = g(f(x))$

$\therefore$ $f(x) = \sqrt[3]{x}$

$g(x) = \dfrac{x}{1+x}$