MA114 - Calculus II
FIRST MIDTERM

Spring Semester
02/10/2004

Name: Answer Key Section #: ____

Sec. #	Instructor	TAs	Lectures	Recitations
001	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 8:00-9:15, CB 347
002	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 12:30-1:45, CP 155
003	A. Corso	S. Petrovic	MWF 8:00-8:50, CP 222	TR 3:30-4:45, CB 347

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

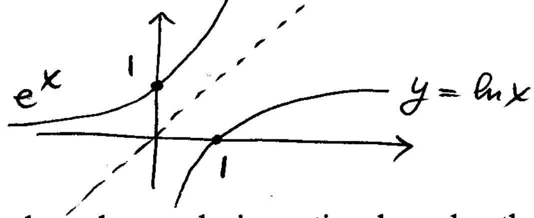
- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

QUESTION	SCORE	TOTAL
1.		15
2.		15
3.		21
4.		24
5.		10
6.		15
Bonus.		5
TOTAL		100

1. (a) (3pts) Define the function e^x from the function $\ln x$.

see the explanation given in class and in the textbook. Essentially ... it is the inverse of $y = \ln x$ Hence, $e^{\ln x} = x$ and $\ln e^x = x$

its graph



(b) (4pts) If a, b are any real numbers and r is a rational number then

* $e^{a+b} =$ $e^a e^b$;
* $e^{a-b} =$ e^a / e^b; $e^{-b} =$ $1/e^b$;
* $(e^a)^r =$ e^{ar}.

(c) * (4pts) Simplify the following expression: $3\ln\sqrt[3]{t^2-1} - \ln(t+1) =$ $\boxed{\ln(t-1)}$

$$3\ln\sqrt[3]{t^2-1} - \ln(t+1) = \ln\left(\sqrt[3]{t^2-1}\right)^3 - \ln(t+1) =$$
$$= \ln(t^2-1) - \ln(t+1) = \ln\left(\frac{t^2-1}{t+1}\right) = \ln(t-1)$$

* (4pts) Solve the following equation. Namely, if

$$\ln\sqrt{e^{e^x}} - \ln(\ln e^e) = 0$$

then $x =$ $\boxed{\ln(2)}$.

$$\ln\sqrt{e^{e^x}} - \ln(\ln e^{e}) = 0 \longleftrightarrow \frac{1}{2}\ln\left(e^{e^x}\right) - \ln\left(e\,\boxed{\ln(e)}\right) =$$

$$\longleftrightarrow \frac{1}{2}e^x \underbrace{\ln(e)}_{1} - \underbrace{\ln(e)}_{1} = 0 \longleftrightarrow \frac{1}{2}e^x - 1 = 0$$

$$\longleftrightarrow e^x = 2 \longleftrightarrow \ln e^x = \ln 2$$

$$\therefore \boxed{x = \ln 2}$$

pts: /15

2. (*a*) (5pts) Find $g'(1) = \underline{\quad 1 \quad}$, where $g(x)$ is the inverse of the function $f(x) = x^3 + x + 1$.

$$g'(a) = \frac{1}{f'(g(a))}$$

Observe that $f'(x) = 3x^2 + 1$

Also $g(f(x)) = x$ and

$g(f(0)) = 0$ But $f(0) = 1$

$\therefore \boxed{g(1) = 0}$

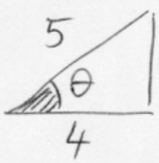
$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{3x^2+1}\Big|_{x=0} = \boxed{1}$$

(*b*) (5pts) Simplify the expressions:

$\sin(\cos^{-1}(4/5)) = \boxed{3/5}$ $\qquad$ $\sin(\tan^{-1} x) = \boxed{x/\sqrt{1+x^2}}$

$\theta = \cos^{-1}(4/5) \Longleftrightarrow \cos\theta = 4/5$

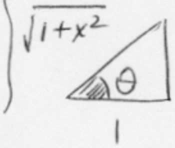
Consider the triangle:



Hence $\sin\theta = 3/5$

$\theta = \tan^{-1} x \Longleftrightarrow \tan\theta = x$

Consider the triangle



$\therefore \sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

(*c*) (5pts) If f is a continuous function such that

$$\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.
(**Hint:** use the Fundamental Theorem of Calculus.)

Differentiate both sides w.r.t. x and use the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}\left[xe^{2x} + \int_0^x e^{-t} f(t)\,dt\right] \Longleftrightarrow$$

$$f(x) = \frac{d}{dx}(xe^{2x}) + e^{-x} f(x) \Longleftrightarrow f(x) = e^{2x} + 2xe^{2x} + e^{-x} f(x)$$

$$\Longleftrightarrow f(x) - e^{-x} f(x) = e^{2x}(1+2x) \Longleftrightarrow f(x)\left[1 - e^{-x}\right] = \frac{e^{2x}(1+2x)}{}$$

Hence $\boxed{f(x) = \dfrac{e^{2x}(1+2x)}{1-e^{-x}} = \dfrac{e^{3x}(1+2x)}{e^x - 1}}$

OR

pts: /15

3. Find the derivative of the following functions. Each problem is worth 7 points.

(a) If $y = \ln(3xe^{-x}) + e^2$ then $y' =$ $\boxed{\dfrac{1-x}{x}}$;

$$y = \ln(3) + \ln(x) + \ln e^{-x} + e^2 = \ln(3) + \ln(x) - x + e^2$$

$$\therefore\ y' = \frac{1}{x} - 1 = \boxed{\frac{1-x}{x}}$$

(b) if $y = x^{\sin x}$ then $\boxed{y' = x^{\sin x}\left[\cos x \cdot \ln x + \dfrac{\sin x}{x}\right]}$ (use logarithmic differentiation);

$$y = x^{\sin x} \iff \ln y = \sin x \cdot \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\sin x \cdot \ln x\right] \iff \frac{1}{y}\frac{dy}{dx} = \cos x \cdot \ln x + \frac{\sin x}{x}$$

use chain rule

$$\therefore\ \frac{dy}{dx} = \overset{x^{\sin x}}{y}\left[\cos x \cdot \ln x + \frac{\sin x}{x}\right]$$

(c) if $y = e^{\tan^{-1}x} + \pi^2$ then $y' =$ $\boxed{e^{\tan^{-1}x} \cdot \dfrac{1}{1+x^2}}$

$$y = e^{\tan^{-1}x} + \pi^2$$

$$y' = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2}$$

pts: /21

4. Find the following integrals. Each problem is worth 6 points.

(a) $\int_0^2 \frac{dt}{8+2t^2} = \boxed{\pi/16}$;

$$= \frac{1}{8}\int_0^2 \frac{dt}{1+\frac{t^2}{4}} = \frac{1}{8}\int_0^2 \frac{dt}{1+\left(\frac{t}{2}\right)^2} \qquad u=\frac{t}{2} \qquad du = \frac{1}{2}dt$$

$$\therefore 2du = dt$$

$$= \frac{1}{8}\int_0^1 \frac{2\,du}{1+u^2} = \frac{1}{4}\int_0^1 \frac{du}{1+t^2} = \frac{1}{4}\tan^{-1}t\Big]_0^1 = \frac{1}{4}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{4}\cdot\frac{\pi}{4}$$

(b) $\int x^{-1}(\ln x)^7\,dx = \boxed{\frac{1}{8}\left[\ln x\right]^8 + C}$;

$$u = \ln x \qquad du = \frac{1}{x}\,dx = x^{-1}dx \qquad \therefore \quad = \int u^7\,du = \frac{1}{8}u^8 + C$$

$$= \frac{1}{8}\left[\ln x\right]^8 + C$$

(c) $\int_0^{\pi/4} \frac{3\sec^2 t}{6+3\tan t}\,dt = \boxed{\ln(3/2)\ ;\ \cong 0.40546}$

$$u = 6 + 3\tan t \qquad du = 3\sec^2 t\,dt$$

$$\therefore \int_6^9 \frac{du}{u} = \ln(u)\Big]_6^9 = \ln(9) - \ln(6) = \ln(9/6) = \ln\left(\frac{3}{2}\right)$$

(d) $\int \sin x\, e^{\cos x}\,dx = \boxed{-e^{\cos x} + C}$.

$$u = \cos x \qquad du = -\sin x\,dx$$

$$\therefore = \int -e^u\,du = -e^u + C = -e^{\cos x} + C$$

pts: /24

One model for the way diseases spread assumes that the rate dy/dt at which the number of infected people changes is proportional to the number of people y. The more the infected people there are, the faster the disease will spread. The fewer there are, the slower it will spread.
We have seen in class that the differential equation

$$\frac{dy}{dt} = ky \qquad y(0) = y_0$$

has solution

$$y = y_0 e^{kt},$$

where k is a constant.

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5. Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1,000?

   (**Hint:** we know that $y(1) = 8{,}000$.)

one year later

$y(t) = 10{,}000\, e^{kt}$ $\qquad$ $y(1) = 8{,}000$

$= (20\%$ less from $10{,}000)$

$\therefore\ 8{,}000 = 10{,}000\, e^{k \cdot 1} = y(1)$

$\therefore\ 4 = 5 e^{k} \iff e^{k} = \frac{4}{5} \quad \therefore\ \boxed{k = \ln(4/5)}$

$y(t) = 10{,}000\, e^{[\ln(4/5)]t} = 10{,}000 \left[e^{\ln(4/5)}\right]^{t} =$

$= 10{,}000 \left(\frac{4}{5}\right)^{t}$

Want to find $\bar{t}$ such that

$1{,}000 = 10{,}000 \left(\frac{4}{5}\right)^{\bar{t}} = y(\bar{t})$

$\therefore\ \frac{1}{10} = \left(\frac{4}{5}\right)^{\bar{t}}$ take ln of both sides

$-\ln(10) = \bar{t} \ln(4/5)$

$$\boxed{\therefore\ \bar{t} = -\frac{\ln(10)}{\ln(4/5)} = \frac{\ln(10)}{\ln(5/4)} \cong 10.31 \text{ years}}$$

pts: /10

6. Use l'Hôpital's rule to find the following limits.

(a) $\lim\limits_{x\longrightarrow 0} \dfrac{x-\sin x}{x^3} = \boxed{1/6}$;

$$\frac{0}{0} \underset{\uparrow \text{ l'Hôpital}}{=} \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{0}{0} \underset{\uparrow \text{ l'Hôpital again}}{=} \lim_{x\to 0} \frac{\sin x}{6x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

(b) $\lim\limits_{x\longrightarrow 0} \dfrac{\sin^{-1}(5x)}{x} = \boxed{5}$;

$\sin^{-1}(0)=0$ so that

$$= \frac{0}{0} \underset{\uparrow \text{ l'Hôpital}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-(5x)^2}} \cdot 5}{1} =$$

$$= \lim_{x\to 0} \frac{5}{\sqrt{1-25x^2}} = \boxed{5}$$

(c) $\lim\limits_{t\longrightarrow \infty} \dfrac{t^2+1}{t\ln t} = \boxed{\infty}$.

$$\frac{\infty}{\infty} \underset{\uparrow \text{ l'Hôpital}}{=} \lim_{t\to\infty} \frac{2t}{\ln(t) + t \cdot \frac{1}{t}} = \lim_{t\to\infty} \frac{2t}{\ln(t)+1}$$

$$= \frac{\infty}{\infty} \underset{\uparrow \text{ l'Hôpital}}{=} \lim_{t\to\infty} \frac{2}{\frac{1}{t}} = \lim_{t\to\infty} 2t = \boxed{\infty}$$

pts: /15

Bonus. Choose one of the following problems.

(*a*) Find the derivative of the following function: $f(x) = \tan^{-1}\sqrt{x^2-1},\ x > 1$. What simpler formula for $f(x)$ can be drawn from your calculation? Explain why?

$$\boxed{f'(x)} = \frac{1}{1+\left(\sqrt{x^2-1}\right)^2} \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x = \frac{1}{x^2} \cdot \frac{x}{\sqrt{x^2-1}} =$$

$$= \boxed{\frac{1}{x\sqrt{x^2-1}}} = \frac{d}{dx}\sec^{-1}x \quad \therefore \boxed{f(x) = \sec^{-1}x + C}$$

But for $\underline{x=1}$ $\quad f(1) = \tan^{-1} 0 = 0 \quad \therefore\ 0 = \underbrace{\sec^{-1} 1}_{=0} + C \quad \therefore \boxed{C=0}$

(*b*) Find the derivative of the following function: $f(x) = (x^2+1)\,\mathrm{sech}\,(\ln x)$. To be precise, show that $f'(x)$ is a constant. Hence, what simpler formula for $f(x)$ can be drawn from your calculation? Explain why?

Observe that $\boxed{\mathrm{sech}(\ln x)} = \dfrac{1}{\cosh(\ln x)} = \dfrac{2}{e^{\ln x} + e^{-\ln x}} =$

$$= \frac{2}{x + x^{-1}} = \frac{2}{x + \frac{1}{x}} = \boxed{\frac{2x}{x^2+1}} \quad \therefore \boxed{f(x) = 2x}$$

$$\boxed{f'(x) = 2}$$

You can do it the long way ......

(*c*) Evaluate the following integral: $\displaystyle\int_1^2 \frac{\cosh(\ln t)}{t}\,dt = \boxed{3/4}$.

set $u = \ln t \quad du = \frac{1}{t}\,dt$

$$= \int_{\ln(1)}^{\ln(2)} \cosh(u)\,du = \int_0^{\ln(2)} \cosh(u)\,du = \sinh(u)\Big]_0^{\ln(2)} =$$

$$= \sinh(\ln(2)) - \cancel{\sinh(0)}^{\,0} = \boxed{3/4}$$

$$\sinh(\ln(2)) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}$$

pts: /5