Review Problems
(Chapter 12)

1. Suppose polar coordinates r and θ are introduced into a function $w = f(x, y)$ to yield $w = f(r, \theta)$. Show that

$$\boxed{\nabla w = \frac{\partial w}{\partial r}\widehat{\mathbf{u}}_{\mathbf{r}} + \frac{\partial w}{\partial \theta}\widehat{\mathbf{u}}_{\theta},}$$

where

$$\widehat{\mathbf{u}}_{\mathbf{r}} = \cos\theta\,\widehat{\mathbf{i}} + \sin\theta\,\widehat{\mathbf{j}} \qquad \widehat{\mathbf{u}}_{\theta} = -\sin\theta\,\widehat{\mathbf{i}} + \cos\theta\,\widehat{\mathbf{j}}$$

are perpendicular unit vectors.

2. Estimate how much wood it takes to make a hollow rectangular box whose inside measurements are 5 ft. long by 3 ft. wide by 2 ft. deep if the box is made of lumber 0.5 in. thick and the box has no top.

3. Suppose that $f(u, v, w)$ is differentiable and that

$$u = x - y \qquad v = y - z \qquad w = z - x;$$

show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

4. Let $U = F(p, V, T)$ be the internal energy of a gas that obeys the ideal gas law (Boyle–Mariotte Law)

$$pV = nRT$$

(n and R constant; p = pressure, V = volume, and T = temperature). Find

$$\left(\frac{\partial U}{\partial p}\right)_V \qquad \left(\frac{\partial U}{\partial T}\right)_V.$$

5. (a) Find equations for the tangent plane and the normal line at the point $P_0(\sqrt{3}/2, 1/2, e^{-1})$ on the surface

$$z - e^{-(x^2+y^2)} = 0.$$

(b) Show that $(0, 0)$ is a point of absolute maximum for the function $z = f(x, y) = e^{-(x^2+y^2)}$. Graph it. Describe the level curves of $z = f(x, y)$.

6. Suppose that the uv-plane is mapped to the xy-plane by setting

$$x = 2uv \qquad y = u^2 - v^2.$$

(a) Show that the images of lines through the origin in the uv-plane are half lines emanating from $(x, y) = (0, 0)$ and that each point of a half line is the image of 2 points;

(b) Show that the image curves of circles $u^2 + v^2 = a^2$ in the uv-plane are circles $x^2 + y^2 = a^4$ in the xy-plane;

(c) Find the determinant of the matrix

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}.$$

7. Find the critical points of

$$f(x, y) = xy + 2x - \ln(x^2 y)$$

in the open first quadrant ($x > 0$, $y > 0$) and show that f takes a minimum there.

8. A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + z^2 = 16$$

enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

9. Find the local maximum and minimum values of the function

$$f(x, y) = x$$

subject to the constraint

$$g(x, y) = 4x^4 + 4y^4 + 8x^2y^2 - 8x^3 - 8xy^2 + 3x^2 - y^2 = 0.$$

10. (a) Show that the previous problem can be formulated in the following equivalent (easier) way: Find the local maximum and minimum values of

$$f(r, \theta) = r \cos\theta$$

subject to the constraint

$$g(r, \theta) = r - \frac{1}{2} - \cos\theta = 0.$$

(Note that r can also be negative).

(b) Can you give a geometric interpretation of this problem? What does $g(r, \theta) = 0$ represent? (Hint: use xmaple's command: "`plot([1/2+ cos(t), t, t=0..2*Pi], coords=polar);`").