MATH 251 FINAL	Spring 96–Sec. 01-02-03 Corso – Higgins	Name: ______________________

PLEASE, BE NEAT AND SHOW ALL YOUR WORK; CIRCLE YOUR ANSWER.

Problem Number	Possible Points	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

WHERE DO YOU SIT?

J														
I														
H														
G														
F														
E														
D														
C														
B														
A														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14

BLACKBOARD

Vector Valued Functions and Motion in Space

$\widehat{\mathbf{T}}$, $\widehat{\mathbf{B}}$, and $\widehat{\mathbf{N}}$: Let $\vec{\mathbf{r}}(t) = x(t)\widehat{\mathbf{i}} + y(t)\widehat{\mathbf{j}} + z(t)\widehat{\mathbf{k}}$, $a \le t \le b$, be the position vector describing the motion in the space of a particle. Then

$$\text{velocity} = \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}, \qquad s = \text{arclength} = \int_a^t |\vec{\mathbf{v}}(u)|\,du, \qquad \text{speed} = \frac{ds}{dt} = |\vec{\mathbf{v}}|,$$

$$\text{acceleration} = \vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2} = \left(\frac{d^2s}{dt^2}\right)\widehat{\mathbf{T}} + \kappa\left(\frac{ds}{dt}\right)^2\widehat{\mathbf{N}} = \frac{d|\vec{\mathbf{v}}|}{dt}\widehat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\widehat{\mathbf{N}},$$

$$\text{unit tangent vector} = \widehat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|},$$

$$\text{principal unit normal vector} = \widehat{\mathbf{N}} = \frac{1}{\kappa}\frac{d\widehat{\mathbf{T}}}{ds} = \frac{d\widehat{\mathbf{T}}/dt}{|d\widehat{\mathbf{T}}/dt|},$$

$$\text{binormal vector} = \widehat{\mathbf{B}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{N}},$$

$$\text{curvature} = \kappa = \left|\frac{d\widehat{\mathbf{T}}}{ds}\right| = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{|\vec{\mathbf{v}}|^3},$$

$$\text{torsion} = \tau = -\left(\frac{d\widehat{\mathbf{B}}}{ds}\right)\cdot\widehat{\mathbf{N}} = \frac{\det\begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{pmatrix}}{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|^2}.$$

Functions of Two or More Variables and Their Derivatives

Gradient and Directional Derivative: Let $f(x, y, z)$ be a function differentiable throughout some region D in the space containing a point $P(x_0, y_0, z_0)$. The **gradient** of f at P is the vector

$$(\nabla f)_P = \frac{\partial f}{\partial x}(P)\hat{\mathbf{i}} + \frac{\partial f}{\partial y}(P)\hat{\mathbf{j}} + \frac{\partial f}{\partial z}(P)\hat{\mathbf{k}}.$$

If $\hat{\mathbf{u}}$ is a unit vector, then the **directional derivative** of f at P in the direction of $\hat{\mathbf{u}}$ is

$$(D_{\hat{\mathbf{u}}} f)_P = (\nabla f)_P \cdot \hat{\mathbf{u}}.$$

Tangent Plane and Normal Line: The eq. of the tangent plane at $P(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ is:

$$\frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0) + \frac{\partial f}{\partial z}(P)(z - z_0) = 0.$$

The eq. of normal line of the surface at P is:

$$x(t) = x_0 + t\,\frac{\partial f}{\partial x}(P), \qquad y(t) = y_0 + t\,\frac{\partial f}{\partial y}(P), \qquad z(t) = z_0 + t\,\frac{\partial f}{\partial z}(P).$$

Critical Points: Let $f(x, y)$ be a continuous function of two independent variables. The points where

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \qquad \text{and the ones where} \qquad \frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{fails to exist}$$

are called **critical points** of $f(x, y)$.

Local Max & Min Test: Let P be a critical point of $f(x, y)$, then

- f has a **local maximum** at P if

$$\frac{\partial^2 f}{\partial x^2}(P) < 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 > 0;$$

- f has a **local minimum** at P if

$$\frac{\partial^2 f}{\partial x^2}(P) > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 > 0;$$

- f has a **saddle point** at P if

$$\frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 < 0;$$

- the **test is inconclusive** if

$$\frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 = 0.$$

Method of Lagrange Multipliers: Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable. To find the local extremal values of f subject to the constraint $g = 0$ one finds the values of x, y, z and λ that simultaneously satisfy the equation

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0.$$

Multiple Integrals

Masses, etc.: If $\delta = \delta(x, y, z)$ is the density function of an object occupying a region D in space, then the mass, the first moments, the center of mass, and the second moments are given by the following formulas:

$$M = \iiint_D \delta\, dV$$

$$M_{yz} = \iiint_D x\delta\, dV \qquad M_{xz} = \iiint_D y\delta\, dV \qquad M_{xy} = \iiint_D z\delta\, dV$$

$$\overline{x} = \frac{M_{yz}}{M} \qquad \overline{y} = \frac{M_{xz}}{M} \qquad \overline{z} = \frac{M_{xy}}{M}$$

$$I_x = \iiint_D (y^2 + z^2)\delta\, dV \qquad I_y = \iiint_D (x^2 + z^2)\delta\, dV \qquad I_z = \iiint_D (x^2 + y^2)\delta\, dV$$

NOTE: similar (even though simpler) formulas hold in the case of an object in the xy-plane.

Jacobians: Suppose that the region G in the uvw-space is transformed one-to-one into the region D in the xyz-space by differentiable equations of the form

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w)$$

under mild assumptions (always met in this exam) ones has that

$$\iiint_D F(x, y, z)\, dxdydz = \iiint_G F(g, h, k)|J|\, dudvdw$$

where

$$J = J(u, v, w) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \text{Jacobian determinant.}$$

NOTE: a similar (even though simpler) formula holds in the case of a one-to-one transformation between the uv-plane into the xy-plane.

Cylindrical Coordinates: The following are selected equations relating cartesian and cylindrical coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z \qquad x^2 + y^2 = r^2.$$

Here $0 \le \theta \le 2\pi$.
The Jacobian of this transformation is $J(r, \theta, z) = r$.

Spherical Coordinates: The following are selected equations relating cartesian and spherical coordinates:

$$x = \rho\sin\varphi\cos\theta, \qquad y = \rho\sin\varphi\sin\theta, \qquad z = \rho\cos\varphi,$$

$$x^2 + y^2 + z^2 = \rho^2, \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

Here $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$.
The Jacobian of this transformation is $J(\rho, \theta, \varphi) = \rho^2 \sin\varphi$.

Formulas for Integration in Vector Fields

Line Integrals: The line integral of a continuous function $f(x,y,z)$ over a space curve C parametrized by $\vec{\mathbf{r}}(t) = g(t)\widehat{\mathbf{i}} + h(t)\widehat{\mathbf{j}} + k(t)\widehat{\mathbf{k}}$, $a \le t \le b$ is given by

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))|\vec{\mathbf{v}}|\,dt.$$

Surface Integrals: If R is the shadow region of a surface S defined by the equation $f(x,y,z) = c$, and $g(x,y,z)$ is a continuous function defined on S, then the surface integral of g over S is given by

$$\iint_S g(x,y,z)\,d\sigma = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f \cdot \widehat{\mathbf{p}}|}\,dA$$

where $\widehat{\mathbf{p}}$ is a unit vector normal to R and $\nabla f \cdot \widehat{\mathbf{p}} \ne 0$.

Green's Theorems: (C bounds R)

If $\vec{\mathbf{F}} = M(x,y)\widehat{\mathbf{i}} + N(x,y)\widehat{\mathbf{j}}$ is a 2-dimensional vector field then:

$$\text{circulation around } C = \oint_C \vec{\mathbf{F}} \cdot \widehat{\mathbf{T}}\,ds = \oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy,$$

$$\text{flux across } C = \oint_C \vec{\mathbf{F}} \cdot \widehat{\mathbf{n}}\,ds = \oint_C M\,dy - N\,dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy,$$

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Divergence: The **divergence** of a vector field $\vec{\mathbf{F}} = M(x,y,z)\widehat{\mathbf{i}} + N(x,y,z)\widehat{\mathbf{j}} + P(x,y,z)\widehat{\mathbf{k}}$ is:

$$\operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

Divergence Theorem: (S bounds D)

The flux of a 3-dimensional vector field $\vec{\mathbf{F}}$ across an oriented surface S in the direction of $\widehat{\mathbf{n}}$ is given by the formula

$$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{n}}\,d\sigma = \iiint_D \nabla \cdot \vec{\mathbf{F}}\,dV.$$

Curl: The **curl** of a 3-dimensional vector field $\vec{\mathbf{F}}$ is:

$$\operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \det\begin{pmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{pmatrix} =$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\widehat{\mathbf{i}} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\widehat{\mathbf{j}} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\widehat{\mathbf{k}}.$$

Stokes's Theorem: (C bounds S)

The counterclockwise circulation of a vector field $\vec{\mathbf{F}}$ around the boundary C of an oriented surface S with respect to the surface's unit normal vector $\widehat{\mathbf{n}}$ is given by

$$\oint_C \vec{\mathbf{F}} \cdot \widehat{\mathbf{T}}\,ds = \oint_C M\,dx + N\,dy + P\,dz = \iint_S (\nabla \times \vec{\mathbf{F}}) \cdot \widehat{\mathbf{n}}\,d\sigma$$

($\vec{\mathbf{F}}$, C, and S must be "good").

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1. (a) (5 points) Find a unit vector parallel to the line

$$\begin{cases} 3x - 6y - 2z = 15 \\ 2x + y - 2z = 5. \end{cases}$$

(b) (5 points) Describe/Sketch the solid region in the space defined by the inequalities

$$x^2 + z^2 \geq 1 \qquad \text{and} \qquad x^2 + y^2 + z^2 \leq 4.$$

$\boxed{\quad /10}$

2. Find the unit tangent vector $\widehat{\mathbf{T}}$, the unit normal vector $\widehat{\mathbf{N}}$, the speed, and the curvature κ at time t for the plane curve

$$\vec{\mathbf{r}}(t) = \cos^3 t\,\hat{\mathbf{\imath}} + \sin^3 t\,\hat{\mathbf{\jmath}} \qquad 0 \le t \le \frac{\pi}{2}.$$

/10

3. Find all maxima and minima of the function

$$f(x, y) = x^2 + y^2 - 2x - 2y$$

in the closed plane region R bounded below by the x-axis, and bounded above by the semicircle $x^2 + y^2 = 8$, $y \geq 0$. (Hint: To study this function on the boundary of the semicircle you may want to use either the method of Lagrange multipliers or polar coordinates.)

$\boxed{\quad /10}$

4. (a) (5 points) Show that the function

$$u(x, y) = \mathrm{e}^{x} \sin y$$

satisfies Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(b) (5 points) Show that the function

$$u(x, t) = \cos(x - at) - \sin(x + at)$$

satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

/10

5. Use polar coordinates to evaluate the double integral

$$\int_0^2 \left(\int_0^{\sqrt{2x-x^2}} \frac{dy}{\sqrt{x^2+y^2}} \right) dx.$$

(Pay <u>a lot</u> of attention in describing in polar coordinates the region over which the integration takes place.)

/10

6. Find the volume and the center of mass of the homogeneous solid D that is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ (a is a constant) and below by the cone $z^2 = x^2 + y^2$. (You may assume $\delta(x, y, z) = 1$).

/10

7. Use the transformation

$$x = \frac{u+v}{2} \qquad y = v$$

to evaluate the integral

$$\int_0^2 \int_{\frac{1}{2}y}^{\frac{1}{2}(y+4)} y^3(2x-y)\mathrm{e}^{(2x-y)^2}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

/10

8. Let D be the solid cylinder bounded by

$$x^2 + y^2 = 4, \quad z = 0, \quad \text{and} \quad z = 4$$

and let $\hat{\mathbf{n}}$ be the outer unit normal to the boundary S of D. If

$$\vec{\mathbf{F}}(x, y, z) = (x^3 + \tan(yz))\,\hat{\mathbf{i}} + (y^3 - e^{xz})\,\hat{\mathbf{j}} + (3z + x^3)\,\hat{\mathbf{k}}$$

find the flux of $\vec{\mathbf{F}}$ across S.

/10

9. Let S be the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$. Let C be the boundary of S.

(a) Compute the circulation of the vector field

$$\vec{\mathbf{F}}(x, y, z) = y\,\widehat{\mathbf{i}} - x\,\widehat{\mathbf{j}} + yz\,\widehat{\mathbf{k}}$$

along C as a line integral.

(b) Verify (using Stokes's Theorem) the answer you found in part (a) by computing the circulation as a surface integral.

$\boxed{\quad /10}$

10. (a) Show that the curl of the two dimensional vector field

$$\vec{\mathbf{F}}(x,y) = \frac{-y}{x^2+y^2}\,\widehat{\mathbf{i}} + \frac{x}{x^2+y^2}\,\widehat{\mathbf{j}}$$

is zero.

(b) The hypotheses of Stokes's Theorem, however, are not satisfied as the circulation of $\vec{\mathbf{F}}$ along the unit circle $C \equiv x^2 + y^2 = 1$ is not zero. Compute

$$\oint_C \vec{\mathbf{F}} \cdot \widehat{\mathbf{T}}\, ds.$$

(c) **Extra credit:** (5 points) What goes wrong?

/10

11. **Extra Credit:** Consider the polar coordinates r and θ as functions of x and y

$$r = r(x, y) \qquad \theta = \theta(x, y)$$

Note that $r^2 = x^2 + y^2$ and $\tan\theta = y/x$. Calculate the gradient of r and the gradient of θ.

/10