Multivariable Calculus

(Mathematics 640:251:Sec. 01-03 — Spring 1996)

1. Consider the vectors $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$. Find
   - $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$;
   - the distance from $P_1(2, 0, -4)$ to $P_0(2, 1, -3)$;
   - $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$;
   - Write down an equation for a line through $(2, 0, -4)$ to $(2, 1, -3)$;
   - Find the area of the triangle with vertices $(0, 0, 0)$, $(2, 0, -4)$, and $(2, 1, -3)$.

2. Describe the set of points in the space whose
   - cylindrical coordinates satisfy
   $$2 \leq r \leq 4 \qquad 0 \leq z \leq 4;$$
   - spherical coordinates satisfy
   $$\rho = 6 \qquad \varphi = \pi/6;$$
   - cartesian coordinates satisfy
   $$1 - x^2 \leq y \leq x^2 - 1.$$

3. Write down an equation for a plane through the points $P(1, 0, 2)$, $Q(2, 0, 3)$, and $R(1, 2, 6)$. What is the distance from this plane to the origin?

4. Consider the curve
   $$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$
   where
   $$x(t) = \int_0^t \sqrt{2 + \tau}\, d\tau \qquad y(t) = \int_0^t \sqrt{3 - \tau}\, d\tau.$$
   - How long is the curve from $t = 0$ to $t = 1$?
   - Find $\widehat{\mathbf{T}}$ at time t.
   - Find $\widehat{\mathbf{N}}$ at time t.

5. Find $\widehat{\mathbf{T}}$, $\widehat{\mathbf{N}}$, $\widehat{\mathbf{B}}$, ds/dt, and κ at time t for the curve
   $$\vec{\mathbf{r}}(t) = (e^t \sin t)\hat{\mathbf{i}} + (e^t \cos t)\hat{\mathbf{j}} + \sqrt{5}e^t\hat{\mathbf{k}}.$$

6. Find the function's domain and range and describe the level curves if
   $$f(x, y) = \ln(x^2 + y^2).$$

7. Let $f(x,y) = x^2y^5 + e^x \sin(y^2)$. Find

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}.$$

8. Let $f(x,y) = x^2y/(x^4 + y^2)$. Compute

$$\lim_{(x,y) \longrightarrow (0,0)} f(x,y)$$

   or show that it does not exist.

9. Find all the solutions of the 2 dimensional Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

   of the form $f(x,y) = ax^2 + bxy + cy^2$.

10. Consider the function

$$f(x,y,z) = x^3 + y^2 \ln z.$$

   Note that $f(1,2,1) = 1$.

   - If you measure $x = 1 \pm 0.02$, $y = 2 \pm 0.01$, and $z = 1 \pm 0.02$ what is the maximum possible error in computing $f(x,y,z) = 1$?
   - Use linearization to estimate $f(1.1, 2.1, 0.9)$.
   - Suppose that

$$\vec{\mathbf{r}}(t) = x(t)\widehat{\mathbf{i}} + y(t)\widehat{\mathbf{j}} + z(t)\widehat{\mathbf{k}}$$

     is moving along a curve. At time $t = t_0$ the position is $\vec{\mathbf{r}}_{\mathbf{0}} = (1,2,1)$ and the velocity vector is $\vec{\mathbf{v}}_{\mathbf{0}} = (2,1,4)$. Use the chain rule to compute

$$\frac{d}{dt}\left(\frac{1}{|\vec{\mathbf{r}}|}\right).$$