MATH 265 - 4/16/97 SECOND MIDTERM	Spring 97–Sec. 01-03 Dr. A. Corso	Name: ______________________ ______________________

PLEASE, BE NEAT AND SHOW ALL YOUR WORK; CIRCLE YOUR ANSWER.

Problem Number	Possible Points	Points Earned
1	15	
2	10	
3	15	
4	10	
5	10	
6	10	
7	15	
8	15	
E.C.	5	
TOTAL	100	

WHERE DO YOU SIT?

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BLACKBOARD

1. Let W be a subspace of the vector space V and suppose that $W = \mathrm{span}\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$, where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent vectors (hence a basis for W).

   (a) Show that the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ defined as

   $$\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_3 \qquad \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{v}_3, \qquad \mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$$

   are also linearly independent.

   (b) Show that every vector $\mathbf{w}$ in W, i.e. a vector of the form $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$, can also be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

   Therefore (a) and (b) show that the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ form a basis for W as well.

grade: /15

2. Mark each of the following True or False:

True ☐	False ☐	Let A be an n by n matrix and suppose that the system of linear equations $A\mathbf{X} = \mathbf{0}$ has only the trivial solution $\mathbf{X} = \mathbf{0}$. Then $A\mathbf{X} = \mathbf{b}$ has a unique solution for every column vector $\mathbf{b}$ with the appropriate number of components.
True ☐	False ☐	Let A be a 6 by 12 matrix whose rank is 6. Then for some vector $\mathbf{b}$ the system $A\mathbf{X} = \mathbf{b}$ has exactly one solution.
True ☐	False ☐	Let V be an inner product space. Every orthogonal set of V is a basis for V.
True ☐	False ☐	Let $\mathbf{v}$ be a vector in $\mathbf{R}^n$ and let W be a subspace of $\mathbf{R}^n$. If the projection of $\mathbf{v}$ onto W is $\mathbf{v}$ itself then $\mathbf{v}$ is in W.
True ☐	False ☐	Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors in $\mathbf{R}^n$ and suppose that they have the same projection onto a non trivial subspace W of $\mathbf{R}^n$. Then $\mathbf{v}_1 = \mathbf{v}_2$.
True ☐	False ☐	Let V be an inner product space. The length, or norm, of every vector $\mathbf{v}$ in V is a positive real number.
True ☐	False ☐	Let V be an inner product space. Every set of linearly independent vectors is also an orthogonal set.
True ☐	False ☐	Let V an inner product space. We have $\| \mathbf{v} + \mathbf{u} \|^2 = \| \mathbf{v} \|^2 + \| \mathbf{u} \|^2$ for all vectors $\mathbf{v}, \mathbf{u}$ in V.
True ☐	False ☐	Let A be an m by n matrix. Then the rowspace of A is perpendicular to the nullspace of A.
True ☐	False ☐	Let A be an n by n matrix and let $\mathbf{v}$ be an eigenvector for A associated to the eigenvalue λ. Then for all $c \in \mathbf{R}$ we have that $c\mathbf{v}$ is also an eigenvector for A associated to the same λ.

grade: /10

3. Let A be the following 3 by 4 matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 2 & -3 \\ 1 & -1 & -3 & 4 \end{bmatrix}$$

and suppose that it is given that

$$\texttt{rref}(A|\mathbf{0}) = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(a) Find an orthonormal basis for the nullspace of A. What is its dimension?

(b) Find an orthonormal basis for the range of A. What is its dimension?

(c) Do your answers agree with the following output from matlab?

$$\texttt{null(A)} = \begin{bmatrix} 1343/1514 & -391/2042 \\ -769/2953 & 89/270 \\ -757/7154 & -862/1081 \\ -167/456 & -196/419 \end{bmatrix} \qquad \texttt{orth(A)} = \begin{bmatrix} 565/1644 & -729/862 \\ -341/618 & 331/1948 \\ 2671/3515 & 560/1107 \end{bmatrix}.$$

If not, carefully explain who is right and who is wrong.

grade: /15

4. (a) Let V be the vector space of all continuous real valued functions defined on the interval $[0,1]$ equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$$

for all $f, g \in V$. Find for which value of the number α the two following functions

$$x(t) = t \qquad y(t) = 1 - \alpha t$$

are perpendicular.

(b) The vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ satisfy the following conditions

$$\| \mathbf{u} \| = 3 \qquad \| \mathbf{v} \| = 5.$$

Find the inner product $\langle \mathbf{u} + 2\mathbf{v}, \mathbf{u} - 2\mathbf{v}\rangle$.

grade: /10

5. The vectors $\mathbf{w}_1 = (2, 1, 0, -1)$, $\mathbf{w}_2 = (1, 1, 1, 0)$ form a basis for the subspace W of $\mathbf{R}^4$.

   (a) Find a basis for the subspace $W^\perp$.

   (b) Find vectors $\mathbf{v}_1 \in W$ and $\mathbf{v}_2 \in W^\perp$ so that the vector $\mathbf{v} = (3, 1, 0, 0)$ can be written as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$.

grade: /10

6. Find the eigenvalues and the eigenvectors of the matrix A if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Make sure to check that the vectors you obtained are indeed eigenvectors.

grade: /10

7. Find the eigenvalues and the eigenvectors of the matrix A if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}.$$

Make sure to check that the vectors you obtained are indeed eigenvectors.

grade: /15

8. Solve the following initial value problem

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

grade: /15

Extra Credit If you have time left, choose one of the following problems:

(a) (5 pts) Let A be an invertible n by n matrix. Suppose that $\mathbf{v}$ is an eigenvector for A associated to the eigenvalue λ. Can λ be equal to 0? Why? Show that $\mathbf{v}$ is also an eigenvector for A^{-1}. Which is the eigenvalue of A^{-1} that corresponds to $\mathbf{v}$?

(b) (5 pts) Let A be an n by n matrix. Suppose that $\mathbf{v}$ is an eigenvector for A associated to the eigenvalue λ. Show that $\mathbf{v}$ is also an eigenvector for A^n. Which is the eigenvalue of A^n that corresponds to $\mathbf{v}$? Recall that A^n means $\underbrace{A \cdot A \cdots A}_{n \text{ times}}$.

grade: /5