Heron's Formula for Triangular Area

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Heron of Alexandria

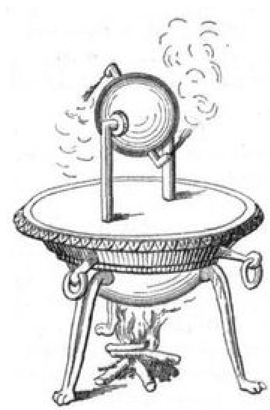
- Physicist, mathematician, and engineer
- Taught at the museum in Alexandria
- Interests were more practical (mechanics, engineering, measurement) than theoretical
- He is placed somewhere around 75 A.D. (±150)

Heron’s Works

- Automata
- Mechanica
- Dioptra
- Metrica
- Pneumatica
- Catoptrica
- Belopoecia
- Geometrica
- Stereometrica
- Mensurae
- Cheirobalistra

The Aeolipile

Heron’s Aeolipile was the first recorded steam engine. It was taken as being a toy but could have possibly caused an industrial revolution 2000 years before the original.



Metrica

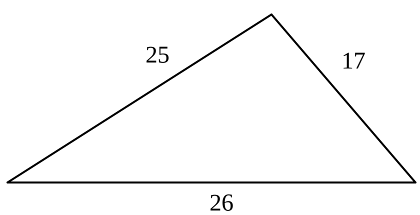
- Mathematicians knew of its existence for years but no traces of it existed
- In 1894 mathematical historian Paul Tannery found a fragment of it in a 13th century Parisian manuscript
- In 1896 R. Schöne found the complete manuscript in Constantinople.
- Proposition I.8 of *Metrica* gives the proof of his formula for the area of a triangle

How is Heron's formula helpful?

How would you find the area of the given triangle using the most common area formula?

$$A = \tfrac{1}{2}bh$$

Since no height is given, it becomes quite difficult…



Heron’s Formula

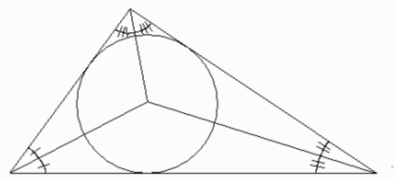
Heron’s formula allows us to find the area of a triangle when only the lengths of the three sides are given. His formula states:

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

Where *a*, *b*, and *c*, are the lengths of the sides and *s* is the semiperimeter of the triangle.

The Preliminaries...

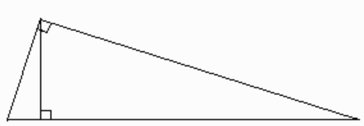
Proposition 1



Proposition IV.4 of Euclid's *Elements*.

The bisectors of the angles of a triangle meet at a point that is the center of the triangles inscribed circle. (Note: this is called the incenter)

Proposition 2

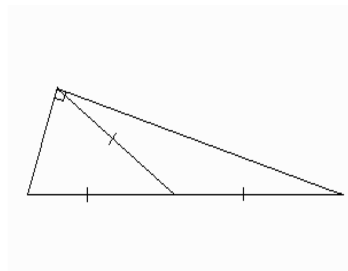


Proposition VI.8 of Euclid's *Elements*.

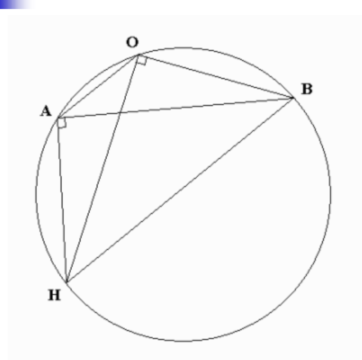
In a right-angled triangle, if a perpendicular is drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle and to one another.

Proposition 3

In a right triangle, the midpoint of the hypotenuse is equidistant from the three vertices.

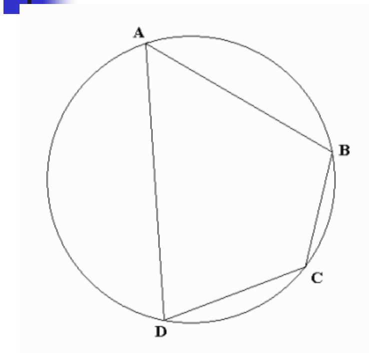


Proposition 4



If *AHBO* is a quadrilateral with diagonals *AB* and *OH*, then if $\angle HOB$ and $\angle HAB$ are right angles (as shown), then a circle can be drawn passing through the vertices *A*, *O*, *B*, and *H*.

Proposition 5



Proposition III.22 of Euclid's *Elements*.

The opposite angles of a cyclic quadrilateral sum to two right angles.

Semiperimeter

The semiperimeter, *s*, of a triangle with sides *a*, *b*, and *c*, is

$$s = \frac{a+b+c}{2}$$

Heron's Proof...

Heron's Proof

- The proof for this theorem is broken into three parts.
- Part A inscribes a circle within a triangle to get a relationship between the triangle's area and semiperimeter.
- Part B uses the same circle inscribed within a triangle in Part A to find the terms s-a, s-b, and s-c in the diagram.
- Part C uses the same diagram with a quadrilateral and the results from Parts A and B to prove Heron's theorem.

Restatement of Heron's Formula

For a triangle having sides of length *a*, *b*, and *c* and area *K*, we have

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

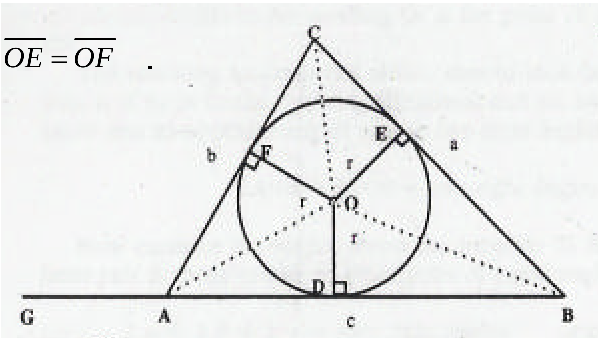
where *s* is the triangle's semiperimeter.

Heron's Proof: Part A

Let ABC be an arbitrary triangle such that side AB is at least as long as the other two sides.

Inscribe a circle with center O and radius r inside of the triangle.

Therefore, $\overline{OD} = \overline{OE} = \overline{OF}$.



Heron's Proof: Part A (cont.)

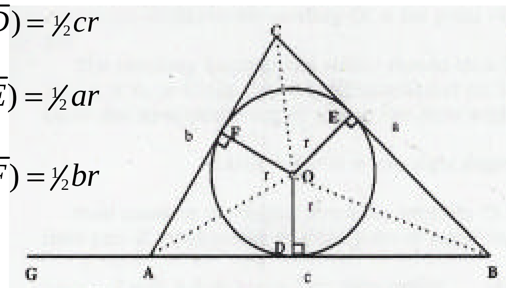
Now, the area for the three triangles ?AOB, ?BOC, and ?COA is found using the formula

½(base)(height).

Area ?AOB $= \frac{1}{2}(\overline{AB})(\overline{OD}) = \frac{1}{2}cr$

Area ?BOC $= \frac{1}{2}(\overline{BC})(\overline{OE}) = \frac{1}{2}ar$

Area ?COA $= \frac{1}{2}(\overline{AC})(\overline{OF}) = \frac{1}{2}br$



Heron's Proof: Part A (cont.)

We know the area of triangle ABC is K. Therefore

$$K = Area(\Delta ABC) = Area(\Delta AOB) + Area(\Delta BOC) + Area(\Delta COA)$$

If the areas calculated for the triangles ?AOB, ?BOC, and ?COA found in the previous slides are substituted into this equation, then K is

$$K = \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br = r\left(\frac{a+b+c}{2}\right) = rs$$

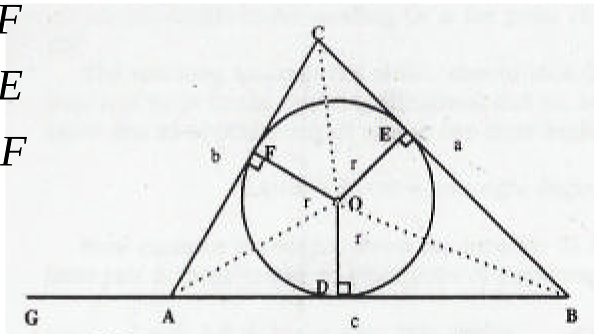
Heron's Proof: Part B

When inscribing the circle inside the triangle ABC, three pairs of congruent triangles are formed (by Euclid's Prop. I.26 AAS).

$$\Delta AOD \cong \Delta AOF$$

$$\Delta BOD \cong \Delta BOE$$

$$\Delta COE \cong \Delta COF$$



Heron's Proof: Part B (cont.)

- Using corresponding parts of similar triangles, the following relationships were found:

$$\overline{AD} = \overline{AF}$$

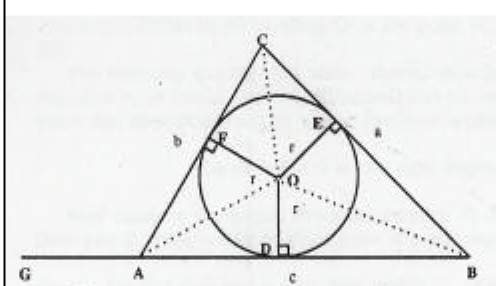
$$\overline{BD} = \overline{BE}$$

$$\overline{CE} = \overline{CF}$$

$$\angle AOD = \angle AOF$$

$$\angle BOD = \angle BOE$$

$$\angle COE = \angle COF$$



Heron's Proof: Part B (cont.)

- The base of the triangle was extended to point G where AG = CE. Therefore, using construction and congruence of a triangle:

$$\overline{BG} = \overline{BD} + \overline{AD} + \overline{AG} = \overline{BD} + \overline{AD} + \overline{CE}$$

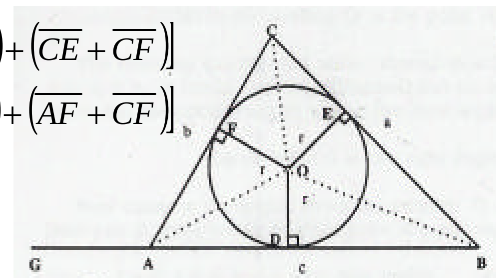
$$\overline{BG} = \tfrac{1}{2}\left(2\overline{BD} + 2\overline{AD} + 2\overline{CE}\right)$$

$$\overline{BG} = \tfrac{1}{2}\left[\left(\overline{BD} + \overline{BE}\right) + \left(\overline{AD} + \overline{AF}\right) + \left(\overline{CE} + \overline{CF}\right)\right]$$

$$\overline{BG} = \tfrac{1}{2}\left[\left(\overline{BD} + \overline{AD}\right) + \left(\overline{BE} + \overline{CE}\right) + \left(\overline{AF} + \overline{CF}\right)\right]$$

$$\overline{BG} = \tfrac{1}{2}\left(\overline{AB} + \overline{BC} + \overline{AC}\right)$$

$$\overline{BG} = \tfrac{1}{2}\left(c + a + b\right) = s$$



Heron's Proof: Part B (cont.)

- Since $\overline{BG} = s$, the semi-perimeter of the triangle is the long segment straighten out. Now, s-c, s-b, and s-a can be found.

$$s - c = \overline{BG} - \overline{AB} = \overline{AG}$$

Since AD = AF and AG = CE = CF,

$$s - b = \overline{BG} - \overline{AC} = \left(\overline{BD} + \overline{AD} + \overline{AG}\right) - \left(\overline{AF} + \overline{CF}\right)$$

$$= \left(\overline{BD} + \overline{AD} + \overline{CE}\right) - \left(\overline{AD} + \overline{CE}\right)$$

$$= \overline{BD}$$

Heron's Proof: Part B (cont.)

Since BD = BF and AG = CE,

$$s-a=\overline{BG}-\overline{BC}=\left(\overline{BD}+\overline{AD}+\overline{AG}\right)-\left(\overline{BE}+\overline{CE}\right)$$
$$=\left(\overline{BD}+\overline{AD}+\overline{CE}\right)-\left(\overline{BD}+\overline{CE}\right)$$
$$=\overline{AD}$$

Heron's Proof: Part B (cont.)

- In Summary, the important things found from this section of the proof.

$$\overline{BG} = \tfrac{1}{2}(c+a+b) = s$$

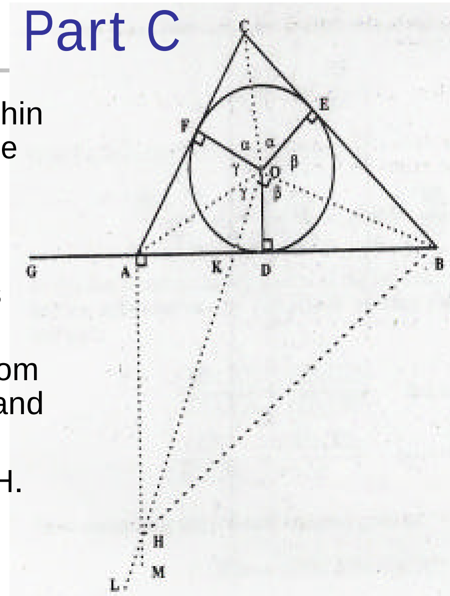
$$s-c=\overline{AG}$$

$$s-b=\overline{BD}$$

$$s-a=\overline{AD}$$

Heron's Proof: Part C

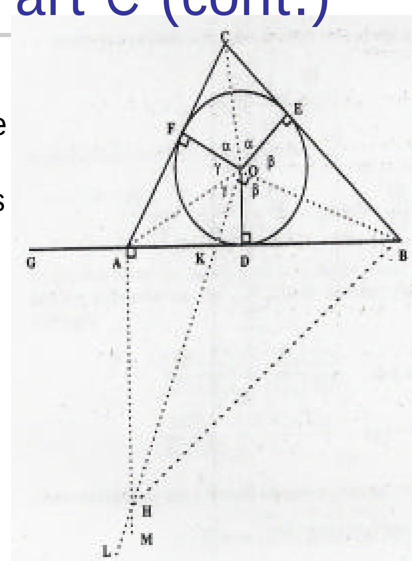
- The same circle inscribed within a triangle is used except three lines are now extended from the diagram.
- The segment OL is drawn perpendicular to OB and cuts AB at point K.
- The segment AM is drawn from point A perpendicular to AB and intersects OL at point H.
- The last segment drawn is BH.
- The quadrilateral AHBO is formed.



Heron's Proof: Part C (cont.)

- Proposition 4 says the quadrilateral AHBO is cyclic while Proposition 5 by Euclid says the sum of its opposite angles equals two right angles.

$$\angle AHB + \angle AOB = 2 \text{ right angles}$$



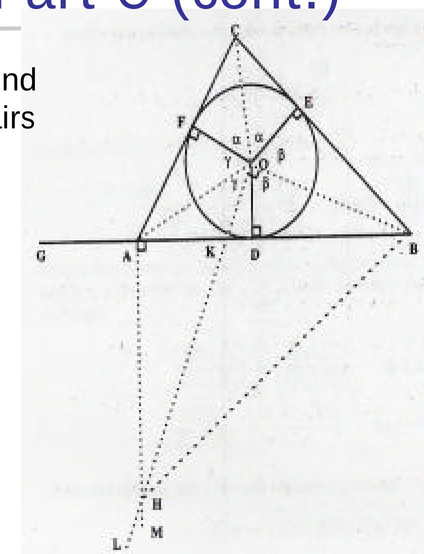
Heron's Proof: Part C (cont.)

- By congruence, the angles around the center O reduce to three pairs of equal angles to give:

$$2a + 2b + 2g = 4 \text{ rt angles}$$

Therefore,

$$a + b + g = 2 \text{ rt angles}$$



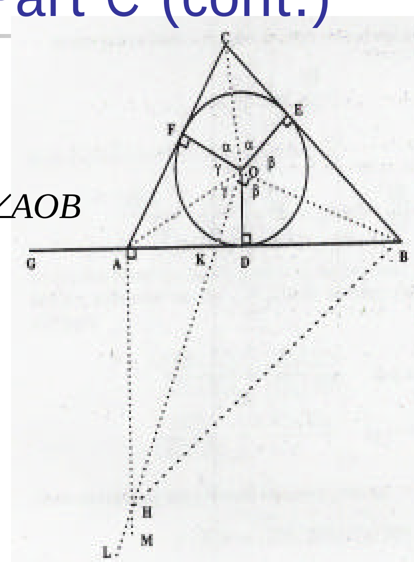
Heron's Proof: Part C (cont.)

- Since $b + a = \angle AOB$, and

$$a + b + g = 2 \text{ rt angles}$$

$$a + \angle AOB = 2 \text{ rt angles} = \angle AHB + \angle AOB$$

Therefore, $a = \angle AHB$.



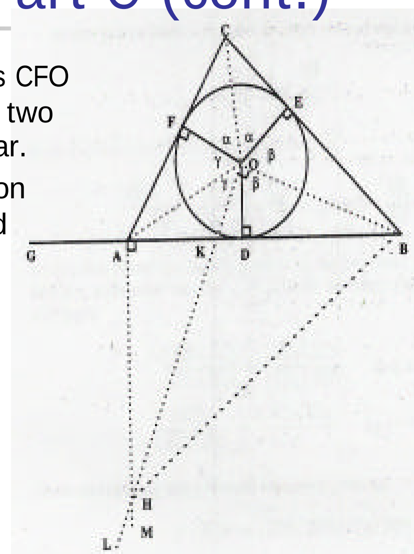
Heron’s Proof: Part C (cont.)

- Since $a = \angle AHB$ and both angles CFO and BAH are right angles, then the two triangles ?COF and ?BHA are similar.
- This leads to the following proportion using from Part B that $\overline{AG} = \overline{CF}$ and $\overline{OH} = r$:

$$\frac{\overline{AB}}{\overline{AH}} = \frac{\overline{CF}}{\overline{OF}} = \frac{\overline{AG}}{r}$$

which is equivalent to the proportion

$$\frac{\overline{AB}}{\overline{AG}} = \frac{\overline{AH}}{r} \qquad (*)$$



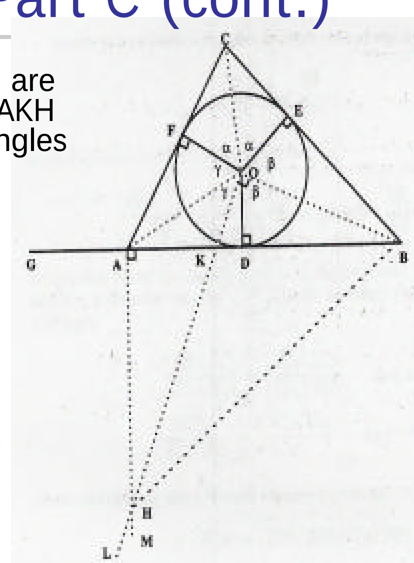
Heron’s Proof: Part C (cont.)

- Since both angles KAH and KDO are right angles and vertical angles AKH and DKO are equal, the two triangles ?KAH and ?KDO are similar.
- This leads to the proportion:

$$\frac{\overline{AH}}{\overline{AK}} = \frac{\overline{OD}}{\overline{KD}} = \frac{r}{\overline{KD}}$$

Which simplifies to

$$\frac{\overline{AH}}{r} = \frac{\overline{AK}}{\overline{KD}} \qquad (**)$$



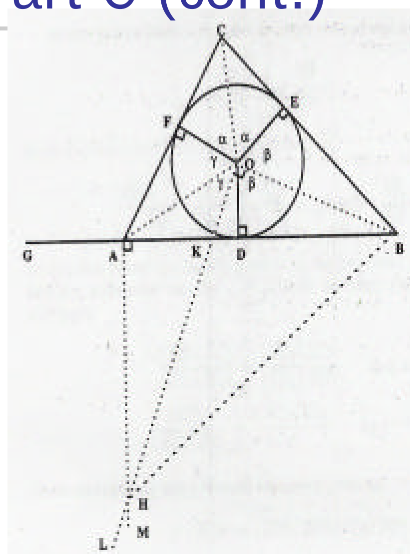
Heron's Proof: Part C (cont.)

- The two equations

$$\frac{\overline{AB}}{\overline{AG}} = \frac{\overline{AH}}{r} \quad (*) \text{ and } \frac{\overline{AH}}{r} = \frac{\overline{AK}}{\overline{KD}} \quad (**)$$

are combined to form the key equation:

$$\frac{\overline{AB}}{\overline{AG}} = \frac{\overline{AK}}{\overline{KD}} \qquad (***)$$



Heron's Proof: Part C (cont.)

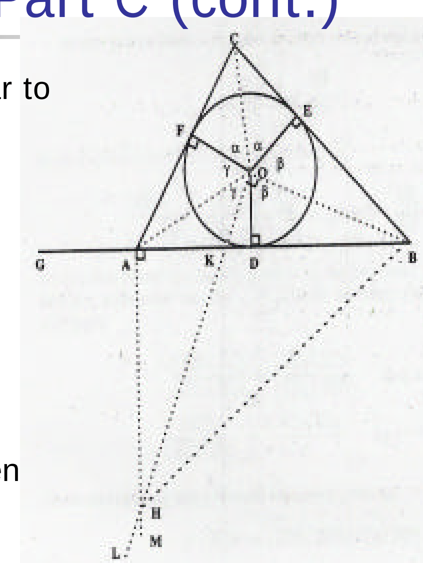
- By Proposition 2, ?KDO is similar to ?ODB where ?BOK has altitude OD=r.
- This gives the equation:

$$\frac{\overline{KD}}{r} = \frac{r}{\overline{BD}}$$

which simplifies to

$$\left(\overline{KD}\right)\left(\overline{BD}\right) = r^2 \qquad (****)$$

(r is the mean proportional between magnitudes KD and BD)



Heron's Proof: Part C (cont.)

- One is added to equation (***), the equation is simplified, then BG/BG is multiplied on the right and BD/BD is multiplied on the left, then simplified.

$$\frac{\overline{AB}}{\overline{AG}} = \frac{\overline{AK}}{\overline{KD}}$$

$$\frac{\overline{AB}}{\overline{AG}} + 1 = \frac{\overline{AK}}{\overline{KD}} + 1$$

$$\frac{\overline{AB} + \overline{AG}}{\overline{AG}} = \frac{\overline{AK} + \overline{KD}}{\overline{KD}}$$

$$\frac{\overline{BG}}{\overline{AG}} = \frac{\overline{AD}}{\overline{KD}}$$

$$\left(\frac{\overline{BG}}{\overline{BG}}\right)\left(\frac{\overline{BG}}{\overline{AG}}\right) = \left(\frac{\overline{AD}}{\overline{KD}}\right)\left(\frac{\overline{BD}}{\overline{BD}}\right)$$

Using the equation $(\overline{KD})(\overline{BD}) = r^2$ (****) this simplifies to:

$$\frac{(\overline{BG})^2}{(\overline{AG})(\overline{BG})} = \frac{(\overline{AD})(\overline{BD})}{r^2}$$

Heron's Proof: Part C (cont.)

- Cross-multiplication of $\frac{(\overline{BG})^2}{(\overline{AG})(\overline{BG})} = \frac{(\overline{AD})(\overline{BD})}{r^2}$ produced

$r^2(\overline{BG})^2 = (\overline{AG})(\overline{BG})(\overline{AD})(\overline{BD})$. Next, the results from Part B are needed. These are:

$$\overline{BG} = s \qquad s - b = \overline{BD}$$

$$s - c = \overline{AG} \qquad s - a = \overline{AD}$$

Heron's Proof: Part C (cont.)

- The results from Part B are substituted into the equation:

$$r^2\left(\overline{BG}\right)^2 = \left(\overline{AG}\right)\left(\overline{BG}\right)\left(\overline{AD}\right)\left(\overline{BD}\right)$$

$$r^2 s^2 = (s-c)(s)(s-b)(s-c)$$

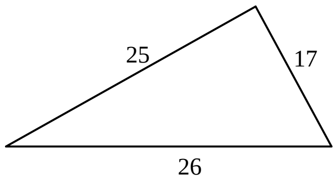
- We know remember from Part A that K=rs, so the equation becomes:

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

- Thus proving Heron's Theorem of Triangular Area

Application of Heron's Formula

We can now use Heron's Formula to find the area of the previously given triangle



$$s = \tfrac{1}{2}(17+25+26) = 34$$

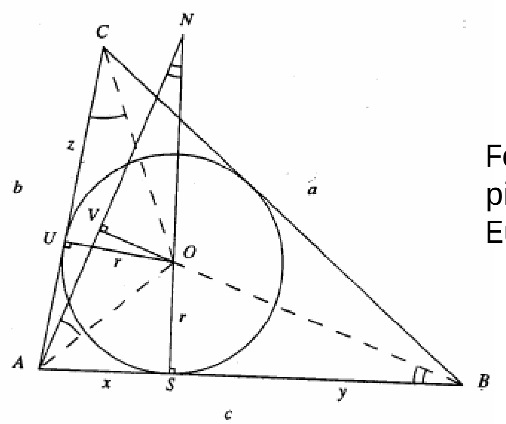
$$K = \sqrt{34(34-17)(34-25)(34-26)} = \sqrt{41616} = 204$$

Euler's Proof of Heron's Formula

Leonhard Euler provided a proof of Heron's Formula in a 1748 paper entitled "Variae demonstrationes geometriae"

His proof is as follows…

Euler's Proof (Picture)



For reference, this is a picture of the proof by Euler.

Euler’s Proof (cont.)

Begin with ΔABC having sides *a*, *b*, and *c* and angles a, b and g

Inscribe a circle within the triangle

Let *O* be the center of the inscribed circle with radius $r = \overline{OS} = \overline{OU}$

From the construction of the incenter, we know that segments *OA*, *OB*, and *OC* bisect the angles of ΔABC with $\angle OAB = \frac{a}{2}$, $\angle OBA = \frac{b}{2}$, and $\angle OCA = \frac{g}{2}$

Euler’s Proof (cont.)

Extend *BO* and construct a perpendicular from *A* intersecting this extended line at *V*

Denote by *N* the intersection of the extensions of segment *AV* and radius *OS*

Because $\angle AOV$ is an exterior angle of $\angle AOB$, observe that

$$\angle AOV = \angle OAB + \angle OBA = \frac{a}{2} + \frac{b}{2}$$

Euler's Proof (cont.)

Because $\angle AOV$ is right, we know that $\angle AOV$ and $\angle OAV$ are complementary

Thus, $\frac{a}{2}+\frac{b}{2}+\angle OAV = 90°$

But $\frac{a}{2}+\frac{b}{2}+\frac{g}{2}=90°$ as well

Therefore, $\angle OAV = \frac{g}{2} = \angle OCU$

Euler's Proof (cont.)

Right triangles ΔOAV and ΔOCU are similar so we get $\overline{AV}/\overline{VO} = \overline{CU}/\overline{OU} = z/r$

Also deduce that ΔNOV and ΔNAS are similar, as are ΔNAS and ΔBAV, as well as ΔNOV and ΔBAV

Hence $\overline{AV}/\overline{AB} = \overline{OV}/\overline{ON}$

This results in $\frac{z}{r} = \frac{\overline{AB}}{\overline{ON}} = \frac{x+y}{\overline{SN}-r}$

So, $z(\overline{SN}) = r(x+y+z) = rs$

Euler's Proof (cont.)

Because they are vertical angles, $\angle BOS$ and $\angle VON$ are congruent, so

$$\angle OBS = 90° - \angle BOS = 90° - \angle VON = \angle ANS$$

ΔNAS and ΔBOS are similar

Hence, $\overline{SN} / \overline{AS} = \overline{BS} / \overline{OS}$

This results in $\overline{SN} / x = y / r$

$$\overline{SN} = (xy) / r$$

Euler's Proof (cont.)

Lastly, Euler concluded that

$$Area(\Delta ABC) = rs = \sqrt{rs(rs)} = \sqrt{z(\overline{SN})(rs)}$$
$$= \sqrt{z\left(\tfrac{xy}{r}\right)rs} = \sqrt{sxyz} = \sqrt{s(s-a)(s-b)(s-c)}$$

Pythagorean Theorem

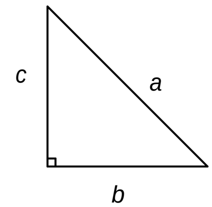
Heron's Formula can be used as a proof of the Pythagorean Theorem

Pythagorean Theorem from Heron's Formula

Suppose we have a right triangle with hypotenuse of length *a*, and legs of length *b* and *c*

The semiperimeter is:

$$s = \frac{a+b+c}{2}$$



Pythagorean Thm. from Heron's Formula (cont.)

$$s-a=\frac{a+b+c}{2}-a=\frac{a+b+c}{2}-\frac{2a}{2}=\frac{-a+b+c}{2}$$

Similarly

$$s-b=\frac{a-b+c}{2} \quad \text{and} \quad s-c=\frac{a+b-c}{2}$$

After applying algebra, we get...

Pythagorean Thm. from Heron's Formula (cont.)

$$\begin{aligned}
&(a+b+c)(-a+b+c)(a-b+c)(a+b-c)\\
&=[(b+c)+a][(b+c)-a][a-(b-c)][a+(b-c)]\\
&=\left[(b+c)^2-a^2\right]\left[a^2-(b-c)^2\right]\\
&=a^2(b+c)^2-(b+c)^2(b-c)^2-a^4+a^2(b-c)^2\\
&=2a^2b^2+2a^2c^2+2b^2c^2-\left(a^4+b^4+c^4\right)
\end{aligned}$$

Pythagorean Thm. from Heron's Formula (cont.)

Returning to Heron's Formula, we get the area of the triangle to be

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}$$
$$= \sqrt{\frac{2a^2b^2+2a^2c^2+2b^2c^2-\left(a^4+b^4+c^4\right)}{16}}$$

Pythagorean Thm. from Heron's Formula (cont.)

Because we know the height of this triangle is c, we can equate our expression to the expression

$$K = \tfrac{1}{2}bh = \tfrac{1}{2}bc$$

Equating both expressions of K and squaring both sides, we get

$$\frac{b^2c^2}{4} = \frac{2a^2b^2+2a^2c^2+2b^2c^2-\left(a^4+b^4+c^4\right)}{16}$$

Cross-multiplication gives us

$$4b^2c^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - \left(a^4 + b^4 + c^4\right)$$

Pythagorean Thm. from Heron's Formula (cont.)

Taking all terms to the left side, we have

$$\left(b^4 + 2b^2c^2 + c^4\right) - 2a^2b^2 - 2a^2c^2 + a^4 = 0$$

$$\left(b^2 + c^2\right)^2 - 2a^2\left(b^2 + c^2\right) + a^4 = 0$$

$$\left[\left(b^2 + c^2\right) - a^2\right]^2 = 0$$

$$\left(b^2 + c^2\right) - a^2 = 0$$

$$a^2 = b^2 + c^2$$

Thus, Heron's formula provides us with another proof of the Pythagorean Theorem