The Bernoullis and The Harmonic Series

By Candice Cprek, Jamie Unseld, and Stephanie Wendschlag

An Exciting Time in Math

- The late 1600s and early 1700s was an exciting time period for mathematics.
- The subject flourished during this period.
- Math challenges were held among philosophers.
- The fundamentals of Calculus were created.
- Several geniuses made their mark on mathematics.

Gottfried Wilhelm Leibniz (1646-1716)

- Described as a universal genius by mastering several different areas of study.
- A child prodigy who studied under his father, a professor of moral philosophy.
- Taught himself Latin and Greek at a young age, while studying the array of books on his father's shelves.
- At age 15 he entered the University of Leipzig, flying through his studies at such a pace that he completed his doctoral dissertation at Altdorf by 20.

Gottfried Wilhelm Leibniz

- He then began work for the Elector of Mainz, a small state when Germany divided, where he handled legal maters.
- In his spare time he designed a calculating machine that would multiply by repeated, rapid additions and divide by rapid subtractions.
- 1672-sent form Germany to Paris as a high level diplomat.

Gottfried Wilhelm Leibniz

- At this time his math training was limited to classical training and he needed a crash course in the current trends and directions it was taking to again master another area.
- When in Paris he met the Dutch scientist named Christiaan Huygens.

Christiaan Huygens

- He had done extensive work on mathematical curves such as the "cycloid".
- This is the path traced by a point fixed to the rim of a circle that is rolling along a horizontal path.
- His discoveries played a role in his design of the first successful pendulum clock.
- He built a reputation on his studies in physics and astronomy.

From Huygens to Leibniz

- With such a wonderful resource, Leibniz was sure to succeed under the guidance of Huygens.
- Huygens suggested for Leibniz to solve the determination of the sum of the reciprocals of the triangular numbers. More simply, numbers of the general form k(k+1)/2.
- Soon he noticed a pattern in the numbers and was able to correctly evaluate the problem by manipulating the fractions.
- Modern mathematicians voice certain reservations about the cavalier manipulations in this argument, but no one can deny the basic ingenuity of his approach.

From Huygens to Leibniz (cont.)

- This was just the beginning of Leibniz's mathematical insights.
- He would soon be applying his great talents to questions Newton had addressed a decade earlier.
- By the time he had left Paris in 1676, he had discovered the fundamental principles of calculus.
- The four years spent in Paris saw him rise from novice to a giant in mathematics.

Meanwhile in England…

- Isaac Newton’s discoveries were known only to a select number of English mathematicians who had seen the handwriting manuscripts kept so quiet.
- 1673-On a visit to London Leibniz was one of the few to see some of these documents and was greatly impressed.
- Leibniz inquired further about the discoveries and Newton responded in two famous 1676 letters: epistola prior and the epistola posterior.
- Leibniz read intently.

The British cry, “FOUL!”

- **1684-Leibniz published his first paper on this amazing new mathematical method in the scholarly journal Acta Eruditorum, which he was the editor of, and his British counterparts cried “foul!”**
- **The paper was titled: “Novo Methodus pro Maximis et Minimis, itemque tangentibus, qua nec fractas, nec irrationals, quantitates moratur, et singulare pro illis calculi genus”--------In English: “A New Method for Maxima and Minima, as well as Tangents, which is impeded neither by Fractional nor Irrational Quantities, and a Remarkable Type of Calculus for this”.**

The British cry, “FOUL!” (cont.)

- The title of this paper was where the subject took its name.
- Thus the world learned the calculus from Leibniz and not Newton.
- The British then accused him of plagiarism from his exchange of letters, visits to England, and the familiarity with the quietly circulating Newtonian manuscript.

Philosophers at War

- The bickering that followed does not constitute one of the more admirable chapters in the history of mathematics.
- At first they tried to stay out of it, but eventually all parties became involved.
- Leibniz freely admitted his contact with Newton but said the information given were only hints at results, not clear-cut methods.
- This form of calculus caught on quickly in Europe, meanwhile Newton still refused to publish anything on the subject.

Philosophers at War

- It was not until 1704 that Newton published a specific account of his method in the appendix to his *Opticks*.
- *De Analysi*, a more through account did not appear in print until 1711.
- Then a full-blown development of Newton’s ideas which was carefully written for learners, appeared only in 1736, nine years after his death.
- So tardy that Leibniz supporters returned the accusation of plagiarism to Newton himself.

The Dust Settles

- Both men deserve credit for independently developing virtually the same body of ideas.
- Not engaged in the controversy, Leibniz devoted his time to the variety of pursuits that had characterized his life.
- He was a major force in the creation of the Berlin Academy, bringing many great thinkers to Europe and putting Berlin on the intellectual map.

The Dust Settles (cont.)

- Leibniz died in 1716. His status had diminished and had never reveled in the same titanic reputation as Newton.
- Where Sir Isaac Newton secluded himself from talented disciples, Leibniz had two of the most enthusiastic followers, Jakob and Johann Bernoulli of Switzerland.

The Bernoulli Family

- One of the most distinguished families in mathematical and scientific history.
- Many mathematical and scientific accomplishments were achieved over several generations of Bernoullis.
- This was a family filled with jealously and rivalry.

Family Reunion (cont…)

- Jakob(1654-1705) and Johann(1667-1748), sons of Nicolaus.
- These brothers are the first Bernoullis to make an impact in mathematical history.
- There existed a bitter sibling rivalry between these two.
- Were among the first to realize the power of calculus.

Jakob Bernoulli

- Was to study philosophy and theology by his parents wishes, but wanted to study mathematics instead, this would be a trend for all the great Bernoullis.
- Graduated with a masters degree in philosophy, but soon turned to mathematics.
- He and his brother Johann after him, were greatly influenced by Leibniz, among others.

Jakob Bernoulli (cont…)

- From 1687 until death, held Chair of Mathematics at Basel University.
- Contributed to the use of Polar Coordinates, discovery of so-called isochrone, and proposed the problem of isoperimetric figures among others.
- Wrote a break through book in probability called Ars conjectandi.

Jakob Bernoulli (cont…)

- Jakob is famous for..
- The Bernoulli distribution and Bernoulli theorem of statistics and probability.
- The Bernoulli numbers and Bernoulli polynomials of number-theory interest.
- The lemniscate of Bernoulli.
  r^2=a cos(2?)
- First used the word integral in a calculus sense in his solution of the isochrone.

Johann Bernoulli

- Was an even bigger contributor to mathematics than Jakob.
- Very jealous and cantankerous
- Jakob taught Johann all he knew.
- A friendly correspondence between the two turn into a bitter rivalry once Johann matched and surpassed Jakob in mathematical knowledge.
- Johann was always jealous of Jakob because he had achieved so much and try very hard to become better.

Johann Bernoulli (cont…)

- Johann was an excellent teacher and exposed the power of calculus to continental Europe.
- Solved a problem his brother posed, problem of the catenary, became his first independent work.
- Later he taught l’Hospital new methods of calculus in exchange for large sums of money.
- L’Hospital’s rule, published in L’Hospital’s book, was really Johann’s solution sent to L’Hospital in a lesson.
- This upset Johann tremendously, but no one believed him when he protested, a proof was later found in 1922 in Basel.
- Johann took over Mathematics Chair at Basel when Jakob died.

Johann Bernoulli (cont…)

- Johann also worked on the problem of the Brachistochrone and tautochrone.
- He proposed the Brachistochrone problem to the entire mathematics world in 1696.
- Not to be outdone Jakob then proposed the isoperimetric problem (minimizing the area enclosed by a curve).
- Johann also made contributions to mechanics with his work in kinetic energy.

Other famous Bernoullis

- Nicolaus(2), son of Nicolaus, brother to Jakob and Johann, mathematician of lesser fame.
- Nicolaus(3), son of Nicolaus(2), nephew of Jakob and Johann.
- Nicolaus(4)(1695-1726), Daniel(1700-1782), and Johann II(1710-1790), sons of Johann.

Other famous Bernoullis

- Nicolaus(4) was a very promising mathematician that proposed the St. Petersburg Paradox.
- He tragically died in St. Petersburg 8 months after arriving and was succeeded by Daniel, his younger brother.

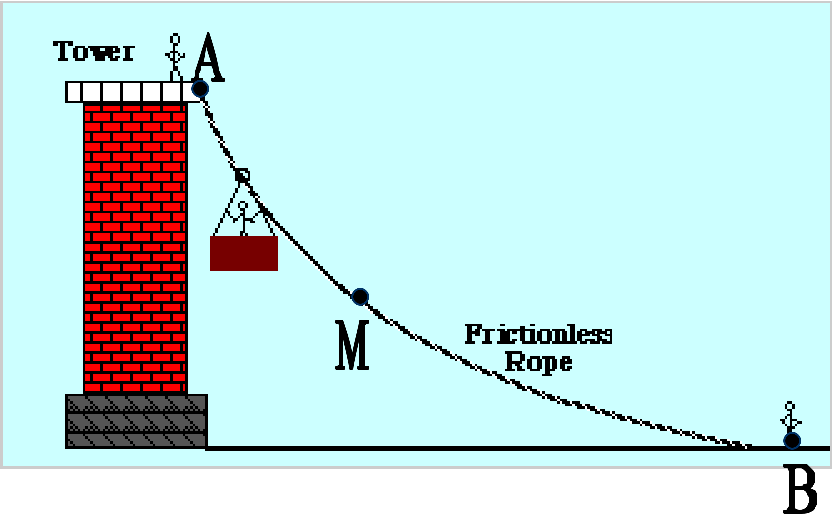
Other famous Bernoullis

- Daniel was the most famous of Johann's sons.
- Johann and Daniel both entered a scientific contest at the University of Paris and they tied.
- Johann was furious he was compared to his offspring and banned him from his house.
- Daniel later published Hydrodynamica, which Johann tried to steal and rename Hydraulica.
- Johann carried a grudge against his son until his death.

Other Famous Bernoullis

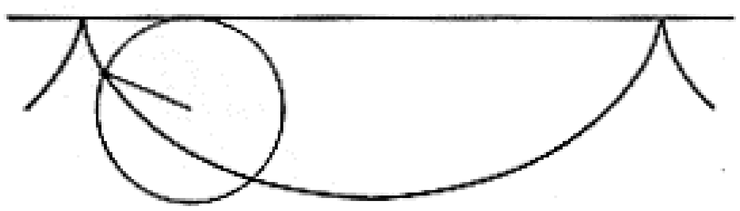
- Johann arranged for Leonard Euler to be sent to St. Petersburg to study with Daniel.
- Euler and Daniel work with mechanics and the study of flexible and elastic bodies.
- Daniel's most prolific work, Hydrodynamica, was based on a single principle, the conservation of energy.
- Developed Bernoulli's equation, a key to describing fluids in motion and fluid dynamics.

The Challenge of the Brachiostochrone



- Points A and B at different heights above ground
- Infinite curves joining A and B
- Imagine object traveling down curve AMB in the shortest time
- Brachistochrone: Greek for “shortest” and “time”

Solution



- Initial guess is straight line because shortest distance
- Johann proposed problem to the mathematical world
- On deadline set by Johann, he received solutions from:
  - Leibniz
  - Jakob
  - Marquis de l'Hospital
  - Anonymous from England (Isaac Newton)

Great Theorem: The Divergence of the Harmonic Series

- Concocted by Johann Bernoulli, but appeared in brother Jakob's book
- Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \text{ is infinite}$$

Johann's Proof

$$A = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ldots + \tfrac{1}{k} + \ldots$$

$$A = \tfrac{1}{2} + \tfrac{2}{6} + \tfrac{3}{12} + \tfrac{4}{20} + \tfrac{5}{30} + \ldots$$

Johann's Proof

$$C = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{12} + \tfrac{1}{20} + \tfrac{1}{30} + \dots = 1$$

$$D = \tfrac{1}{6} + \tfrac{1}{12} + \tfrac{1}{20} + \tfrac{1}{30} + \dots = C - \tfrac{1}{2} = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$E = \tfrac{1}{12} + \tfrac{1}{20} + \tfrac{1}{30} + \dots = D - \tfrac{1}{6} = \tfrac{1}{2} - \tfrac{1}{6} = \tfrac{1}{3}$$

$$F = \tfrac{1}{20} + \tfrac{1}{30} + \dots = E - \tfrac{1}{12} = \tfrac{1}{3} - \tfrac{1}{12} = \tfrac{1}{4}$$

$$G = \tfrac{1}{30} + \dots = F - \tfrac{1}{20} = \tfrac{1}{4} - \tfrac{1}{20} = \tfrac{1}{5}$$

Johann's Proof

Sum down two leftmost columns:

$$C + D + E + F + \ldots$$
$$= \tfrac{1}{2} + \left(\tfrac{1}{6} + \tfrac{1}{6}\right) + \left(\tfrac{1}{12} + \tfrac{1}{12} + \tfrac{1}{12}\right) + \left(\tfrac{1}{20} + \tfrac{1}{20} + \tfrac{1}{20} + \tfrac{1}{20}\right) + \ldots$$
$$= \tfrac{1}{2} + \tfrac{2}{6} + \tfrac{3}{12} + \tfrac{4}{20} + \ldots = A$$

Sum down leftmost and rightmost columns:

$$C + D + E + F + G + \ldots = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ldots = 1 + A$$

Therefore…

$$C + D + E + F + G + \ldots = A = 1 + A$$

"The whole equals the part" so A must be infinite

Previous Proofs of Harmonic Series Divergence

- Earliest-Nicole Oresme (1323-1382)
- Verbal
  - "…add to a magnitude of 1 foot: $^1/_2$,$^1/_3$,$^1/_4$ foot, etc.; the sum of which is infinite. In fact, it is possible to form an infinite number of groups of terms with a sum greater than $^1/_2$. Thus: $^1/_3+^1/_4$ is greater than $^1/_2$; $^1/_5+^1/_6+^1/_7+^1/_8$ is greater than $^1/_2$; $^1/_9+^1/_{10}+\ldots+^1/_{16}$ is greater than $^1/_2$, etc."

Oresme's Proof

- Replace groups of fractions in the harmonic series by smaller fractions that sum to $^1/_2$

$$1+\tfrac{1}{2} > \tfrac{1}{2}+\tfrac{1}{2} = 1$$

$$1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right) > 1+\left(\tfrac{1}{4}+\tfrac{1}{4}\right) = \tfrac{3}{2}$$

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\left(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\right) > \tfrac{3}{2}+\left(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}\right) = \tfrac{4}{2}$$

$$1+\tfrac{1}{2}+\ldots+\tfrac{1}{8}+\left(\tfrac{1}{9}+\tfrac{1}{10}+\ldots+\tfrac{1}{16}\right) > \tfrac{4}{2}+\left(\tfrac{1}{16}+\ldots+\tfrac{1}{16}\right) = \tfrac{5}{2}$$

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k} > \frac{k+1}{2}$$

Oresme's Proof

$$\text{For k} = 9$$

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{512}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^9}>\frac{9+1}{2}=5$$

$$\text{For k} = 99$$

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^9}>\frac{99+1}{2}=50$$

$$\text{For k} = 9999$$

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^{9999}}>\frac{9999+1}{2}=5000$$

Therefore, harmonic series exceeds any finite quantity.

Previous Proofs of Harmonic Series Divergence

- Italian Pietro Mengoli (1625-1686)
- Proof dates back to 1647

Mengoli's Proof

- Preliminary Result:

$$\text{If } a > 1, \text{ then } \frac{1}{a\text{-}1} + \frac{1}{a} + \frac{1}{a+1} > \frac{3}{a}$$

- Proof:

$$2a^3 > 2a^3 - 2a = 2a(a^2 - 1)$$

$$\frac{2a^3}{a^2(a^2-1)} > \frac{2a(a^2-1)}{a^2(a^2-1)}$$

$$\frac{2a}{a^2-1} > \frac{2}{a}$$

Mengoli's Proof

$$\frac{1}{a-1}+\frac{1}{a}+\frac{1}{a+1}=\frac{1}{a}+\left(\frac{1}{a-1}+\frac{1}{a+1}\right)$$

$$=\frac{1}{a}+\frac{2a}{a^2-1}$$ by a bit of algebra

$$>\frac{1}{a}+\frac{2}{a}$$ by the inequality above

$$=\frac{3}{a}$$

Mengoli's Proof

$$\begin{aligned}
H &= 1 + \left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7}\right) + \left(\tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10}\right) + \left(\tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13}\right) + \dots \\
&> 1 + \left(\tfrac{3}{3}\right) + \left(\tfrac{3}{6}\right) + \left(\tfrac{3}{9}\right) + \left(\tfrac{3}{12}\right) + \left(\tfrac{3}{15}\right) + \dots \\
&= 1 + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \dots \\
&= 2 + \left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7}\right) + \left(\tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10}\right) + \left(\tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13}\right) + \dots \\
&> 2 + \left(\tfrac{3}{3}\right) + \left(\tfrac{3}{6}\right) + \left(\tfrac{3}{9}\right) + \left(\tfrac{3}{12}\right) + \left(\tfrac{3}{15}\right) + \dots \\
&= 2 + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \dots \\
&= 3 + \left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7}\right) + \left(\tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10}\right) + \left(\tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13}\right) + \dots
\end{aligned}$$

and so on.