Hippocrates' Quadrature of the Lune (ca. 440 BC)

presentation by

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MA 330 — History of Mathematics

From simple to complex and intricate

- Ancient Greeks were enthralled by the symmetries, the visual beauty, and the logical structure of geometry.

- Particularly intriguing was the manner in which the simple and elementary could serve as foundation for the complex and intricate.

- This enchantment with building the complex from the simple was also evident in the Greeks' geometric constructions.

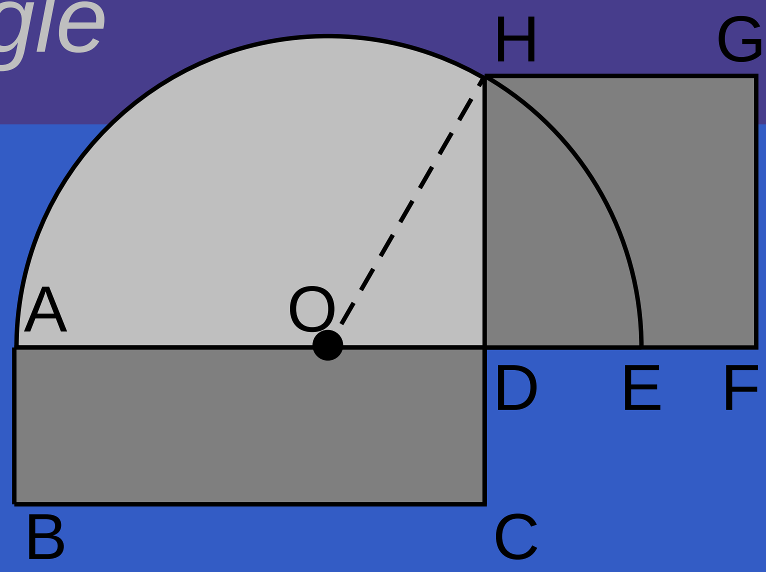
Rules of the "game"!

- The rules of the game required that all constructions be done only with compass and (unmarked) straightedge
- These two fairly unsophisticated tools—allowing the geometer to produce the most perfect, uniform one-dimensional figure (the straight line) and the most perfect, uniform two-dimensional figure (the circle)—must have appealed to the Greek sensibilities for order, simplicity and beauty.
- Moreover, these constructions were within reach of the technology of the day.

The notion of quadrature

- These seemingly unsophisticated tools can produce a rich set of constructions (from the bisection of lines and angles, to the drawing of parallels and perpendiculars, to the creation of regular polygons of great beauty).
- A considerably more challenging problem in ancient Greece was that of the quadrature of a plane figure.
- The quadrature (or squaring) of a plane figure is the construction—using only compass and straightedge—of a square having area equal to that of the original figure. If this is the case, the figure is said to be quadrable (or squarable).

Quadrature of the rectangle



Let ABCD an arbitrary rectangle.
We must construct, with compass and straightedge only, a square having area equal to that of ABCD

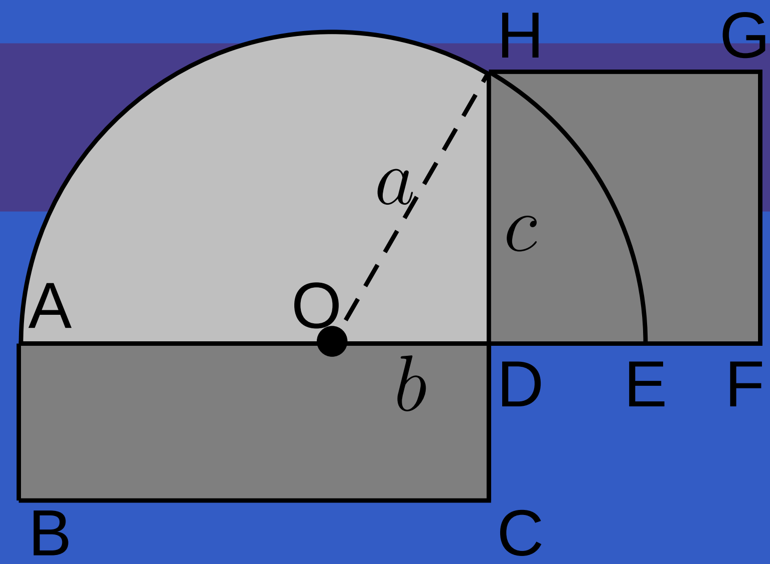
Extend line AD to the right, and use the compass to mark off segment DE with length equal to that of CD.

Bisect AE at O, and with center O and radius $\overline{AO} = \overline{EO}$, describe a semicircle as shown.

At D, construct line DH perpendicular to AE, where H is the point of intersection of the perpendicular and the semicircle.

Construct the “desired” square ... DFGH.

Proof of claim



Why does the square DFGH have the same area as the rectangle ABCD?

Set a, b, c to be the lengths of segments OH, OD and DH.

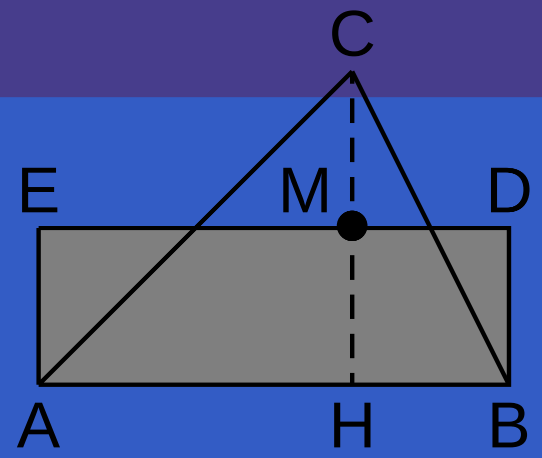
Pythagoras theorem gives us that $a^2 - b^2 = c^2$.

Observe that: $\overline{\text{DE}} = \overline{\text{CD}} = a - b$ and $\overline{\text{AD}} = a + b$.

It follows that:

$$\begin{aligned} \text{Area (rectangle ABCD)} &= \overline{\text{AD}} \times \overline{\text{CD}} \\ &= (a+b)(a-b) = a^2 - b^2 \\ &= c^2 = \text{Area (square DFGH)} \end{aligned}$$

Quadrature of the triangle



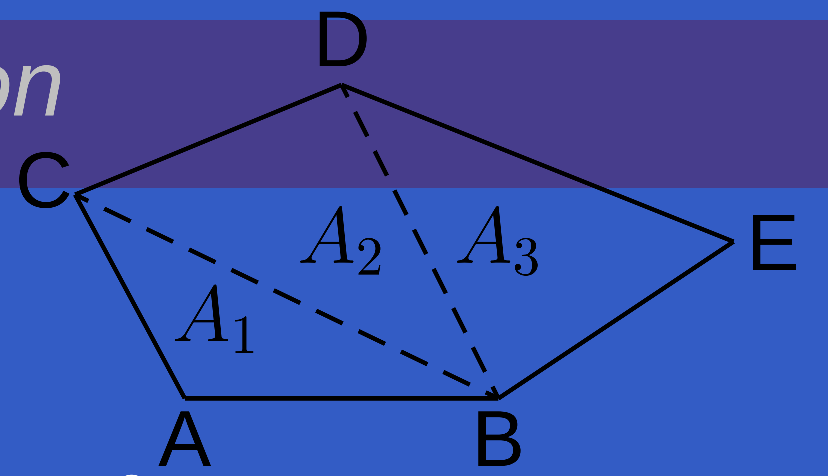
Given a triangle ABC, construct a perpendicular from C meeting AB at point H.

We know that the area of the triangle ABC is $\frac{1}{2}\,\overline{\text{AB}} \times \overline{\text{CH}}$.

If we bisect CH at M and construct a rectangle ABDE with $\overline{\text{DB}} = \overline{\text{EA}} = \overline{\text{MH}}$, we obtain a rectangle with the same area as the triangle ABC.

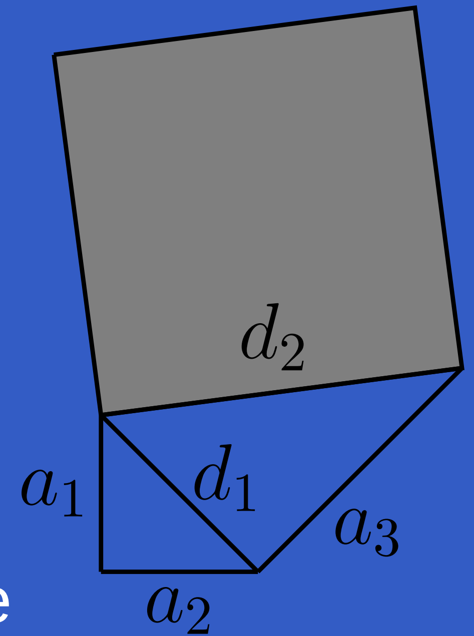
But we already have seen how to square a rectangle.

Quadrature of the polygon



Given a general polygon we can subdivide it into a collection of n triangles, by drawing diagonals (eg n=3).

Now, triangles are quadrable.



We can construct squares with sides a_1, a_2, a_3 and areas A_1, A_2, A_3.

Construct a right triangle with legs a_1, a_2 and hypotenuse d_1. Next construct a triangle with legs d_1, a_3 and hypotenuse d_2. We have:

$$d_2^2 = d_1^2 + a_3^2 = (a_1^2 + a_2^2) + a_3^2 = A_1 + A_2 + A_3.$$

Quadrature of the polygon (cont.ed)

- Obviously, this procedure can be adapted to the situation in which the polygon is divided into any number of triangles.

- By analogous techniques, we could likewise square a figure whose area is the <u>difference</u> between two squarable areas.

Rectilinear vs curvilinear figures

- With the previous techniques, the Greeks of the 5th century BC could square wildly irregular polygons.
- But this triumph was tempered by the fact that such figures are <u>rectilinear</u>.
- What about the issue of whether figures with curved boundaries (<u>curvilinear figures</u>) were also quadrable?
- Initially, this must have seemed unlikely, for there is no obvious means to straighten out curved lines with compass and straightedge.

Three famous problems from antiquity

- The trisection of the angle; that is, the problem of dividing a given angle into three equal parts.

- The duplication of the cube; that is, to find the side of a cube of which the volume is twice that of a given cube (the so-called Delian problem).

- The quadrature of the circle; that is, to find the square of an area equal to that of a given circle.

Hippocrates of Chios (470-410 BC)

- While a talented geometer, Aristotele wrote of him that he "...seemed in other respects to have been stupid and lacking in sense." Legend has it that Hippocrates earned his reputation after being defrauded of his fortune by pirates, who apparently took him for an easy mark. Needing to make a financial recovery, he traveled to Athens and began teaching.

- He is remembered for two important contributions:
  - His composition of the first *Elements*.
  - His quadrature of the lune.

Hippocrates' Elements

- He is credited ($\equiv$since nothing remains today!) with writing the first *Elements*, that is, the first exposition developing the theorems of geometry precisely and logically from a few given axioms or postulates.

- Whatever merits his book had were to be eclipsed, over a century later, by the brilliant *Elements* of Euclid, which essentially rendered Hippocrates' writing obsolete.

- Still, there is reason to believe that Euclid borrowed from his predecessor, and thus we owe much to Hippocrates for his great, if lost, treatise.

Hippocrates' lune

- It must have been quite unexpected when Hippocrates of Chios succeeded (ca. 440 BC) in squaring a curvilinear figure known as a "lune."

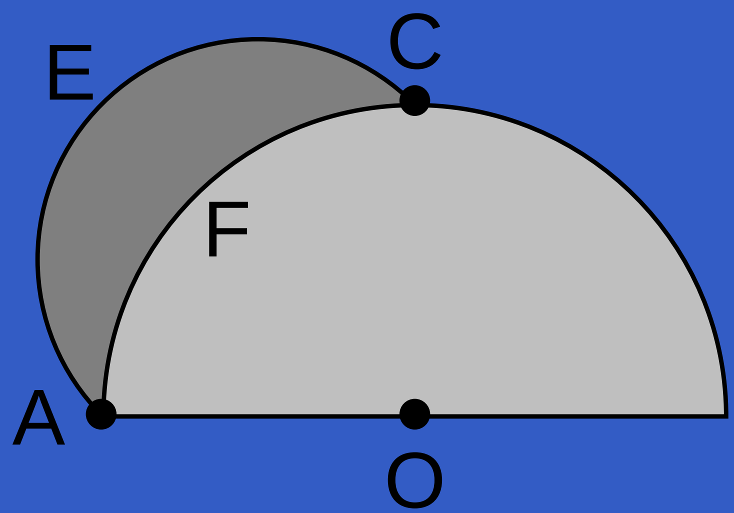
- We do not have Hippocrates' own work, but Eudemus' account of it from around 335 BC. Even here the situation is murky, because we do not really have Eudemus' account either.

- Rather, we have a summary by Simplicius from 530 AD that discussed the writings of Eudemus.

What is a lune?

... A lune is a plane figure bounded by two circular arcs.

For instance:



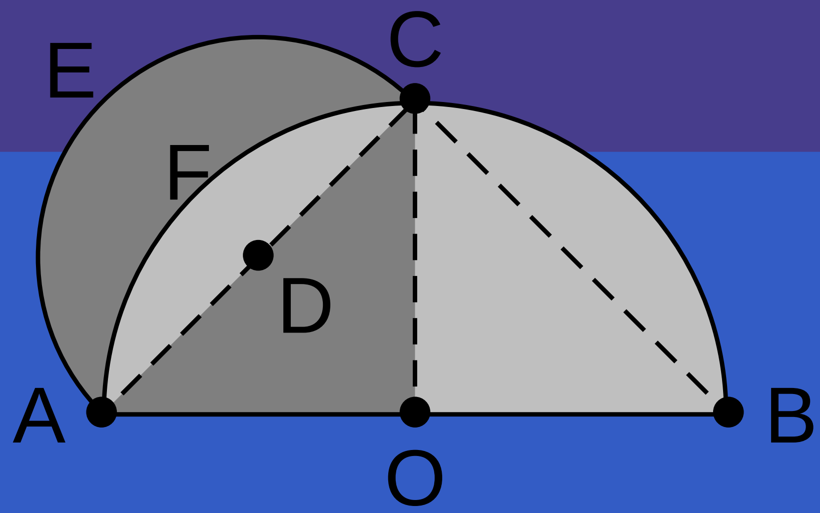
Comments (1)

- Hippocrates did not square all such figures but a particular lune he had carefully constructed (the one in the previous slide!).
- His argument rested upon three preliminary results:
  - The Pythagorean Theorem
  - An angle inscribed in a semicircle is right.
  - The areas of 2 circles or semicircles are to each other as the squares of their diameters.

Comments (2)

- The first two of these results were well known long before Hippocrates. The last proposition, on the other hand, is considerably more sophisticated.
- There is widespread doubt that Hippocrates actually had a valid proof. He may well have thought he could prove it, but modern scholars generally feel that this theorem presented logical difficulties far beyond what Hippocrates would have been to handle.
- This latter result appeared as *Proposition 2* in `Book XII` of Euclid's *Elements*.

Hippocrates' Theorem



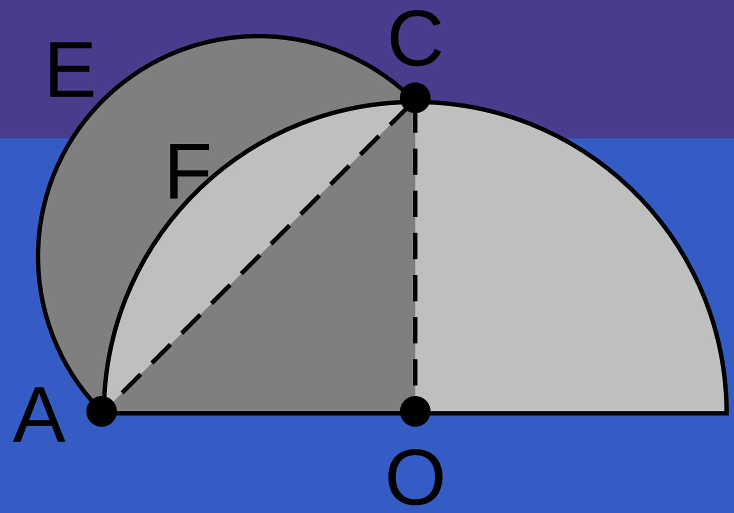
Begin with a semicircle having center O and radius $\overline{AO} = \overline{OB}$.

Construct OC perpendicular to AB, with point C on the semicircle, and draw lines AC and BC.

Bisect AC at D, and using $\overline{AD}$ as a radius and D as center draw semicircle AEC, thus creating lune AECF, which is shaded in the diagram.

Theorem: The lune AECF is quadrable.

Strategy of proof

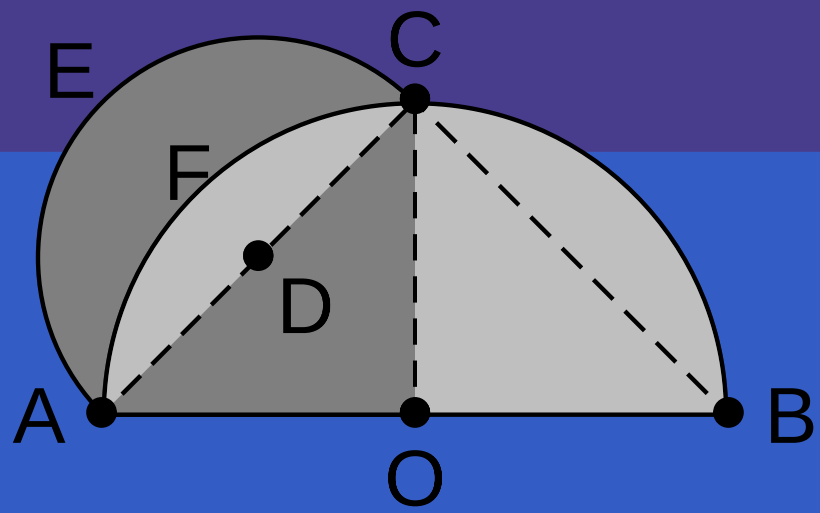


Hippocrates' plan of attack was simple yet brilliant.

He first had to establish that the lune in question had precisely the same area as the shaded triangle AOC.

With this behind him he could then apply the known fact that triangles can be squared to conclude that the lune can be squared as well.

The proof



Note that the angle ACB is right since it is inscribed in a semicircle.

Triangles AOC and BOC are congruent, and so $\overline{\mathrm{AC}} = \overline{\mathrm{BC}}$.

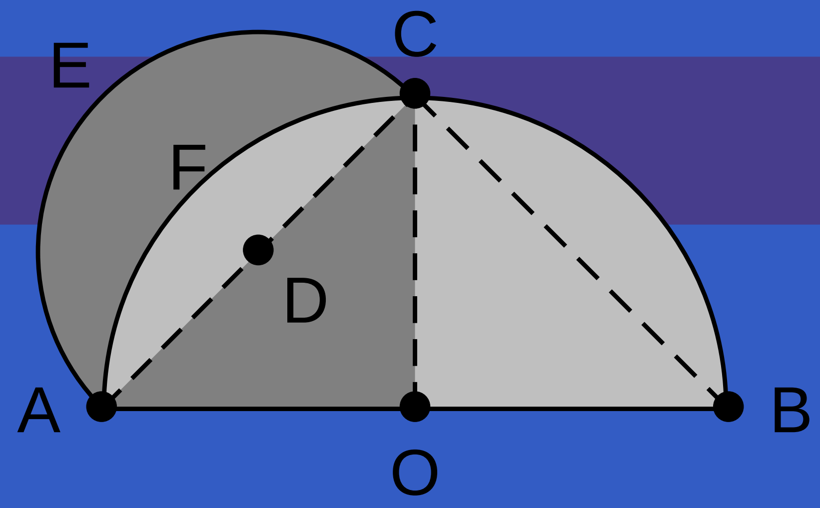
We thus apply the Pythagorean theorem to get

$$\overline{\mathrm{AB}}^2 = \overline{\mathrm{AC}}^2 + \overline{\mathrm{BC}}^2 = 2\,\overline{\mathrm{AC}}^2$$

Then, the semicircle AEC has $\frac{1}{2}$ area of the semicircle ACB:

$$\frac{\text{Area (semicircle AEC)}}{\text{Area (semicircle ACB)}} = \frac{\overline{\mathrm{AC}}^2}{\overline{\mathrm{AB}}^2} = \frac{\overline{\mathrm{AC}}^2}{2\,\overline{\mathrm{AC}}^2} = \frac{1}{2}.$$

The proof (conclusion)



Thus, semicircle AEC has half area of semicircle ACB.

Now, quadrant AFCO has $\frac{1}{2}$ the area of semicircle ACB. So:

Area (semicircle AEC) = Area (quadrant AFCO).

Subtractring the area of their shared region AFCD leaves:

Area (semicircle AEC) - Area (region AFCD)

= Area (quadrant AFCO) - Area (region AFCD)

or

Area (lune AECF) = Area (triangle ACO)

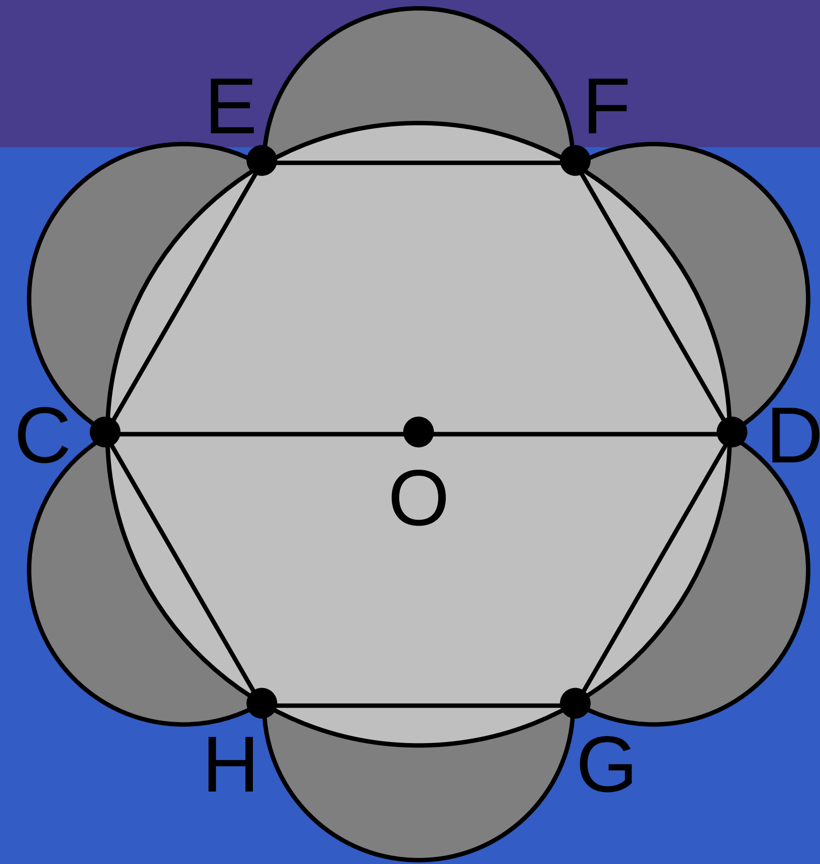
Epilogue

With Hippocrates' success at squaring the lune, Greek mathematicians must have been optimistic about squaring the most perfect curvilinear figure, the circle.

The ancients devoted much time to this problem, and some writers attributed an attempt to Hippocrates himself.

Piecing together the evidence, we gather that what follows is the sort of argument some ancient writers had in mind.

False argument



Imagine an arbitrary circle with diameter AB. Construct a large circle with center O and diameter CD that is twice AB.

Within the larger circle, inscribe a regular hexagon.

Using the six segments CE=EF=FD=DG=GH=CH, construct the six semicircles shown in the figure. This generates the shaded region composed of the six lunes.

False argument (cont.ed)

Now observe that:

Area (hexagon) + 3 Area (circle on AB)

= Area (large circle) + Area (6 lunes)

The large circle, having twice the diameter (of AB), must have 4 times the area of the smaller circle. Hence:

Area (hexagon) + 3 Area (circle on AB)

= 4 Area (circle on AB) + Area (6 lunes)

and subtracting “3 Area (circle on AB)” from both sides of the above equation, we get

Area (circle on AB) = Area (hexagon) - Area (6 lunes)

False argument (end)

Since both the hexagon (being a polygon) and the lunes can be squared, thus the circle on AB can be squared by the simple process of subtracting areas.

Unfortunately, there is a glaring argument: the lunes that Hippocrates squared were not constructed along the side of an hexagon but rather along the side of a square.

The problem of squaring the circle remained unresolved for about 2,000 years.

Ferdinand Lindemann and more

At last in 1882, the German mathematician Ferdinand Lindemann proved that the quadrature of the circle is impossible.

It sufficed to prove that π is a “transcendental number,” that is π is not a solution of any polynomial equation with integer coefficients. (A very hard proof!)

Overall, there are 5 type of lunes that are quadrable: 3 types were found by Hippocrates and 2 more kinds were found by Leonhard Euler in 1771.

In the 20th century Tschebatorew and Dorodnow proved that these 5 are the only quadrable lunes!

Possible ideas for the final project

- The 5 lunes of Hippocrates and Euler.
- The three famous problems of antiquity and their solution: Constructible numbers.
- Lindemann’s proof of the transcendence of π.