Practice Test Date: August 1, 1997
Chapters 8, 9
Final Exam: August 6, 1997; room: EE 170; time: 3:20-5:20 pm
Recall that the final exam is comprehensive.

Linear Algebra for Engineering and Science

(Mathematics 511 – Summer 1997)

Problem #1: For each of the following, decide if the statement is '*always true*' or '*always false*' or '*sometimes true, sometimes false*' when the given condition is true.

1. **Given:** $D = D'$, $D\mathbf{u} = 3\mathbf{u}$ and $D\mathbf{w} = 2\mathbf{w}$.
   Statement: $\langle \mathbf{u}, \mathbf{w} \rangle = 0$.

   always true always false sometimes true, sometimes false

2. **Given:** E is a 4×4 matrix with eigenvalues -2, 0, 1, and 3, G is invertible, and $F = G^{-1}EG$.
   Statement: 1 is an eigenvalue of F.

   always true always false sometimes true, sometimes false

3. **Given:** E is a 4×4 matrix with eigenvalues -2, 0, 1, and 3 and $H = 2E^2 - E - I$.
   Statement: 1 is an eigenvalue of H.

   always true always false sometimes true, sometimes false

4. **Given:** K is a 6×6 matrix with $K^4 = \mathbf{0}$ (matrix).
   Statement: $K = \mathbf{0}$ (matrix).

   always true always false sometimes true, sometimes false

Problem #2: If A and B are $n \times n$ normal matrices, does it follow that AB is also a normal matrix? Justify your answer.

Problem #3: Suppose A is an $n \times n$ real matrix. If A is normal and if the eigenvalues of A are real, does it follow that A is symmetric? Justify your answer.

Problem #4: If A is an $n \times n$ complex matrix, is A similar to its transpose A^t? Justify your answer.

Problem #5: If A is an $n \times n$ normal matrix and $A^3 = A^2$ is it true that $A^2 = A$? Justify your answer.

Problem #6: Verify that the matrix

$$A = \begin{bmatrix} 7 & -16 & -8 \\ -16 & 7 & 8 \\ -8 & 8 & -5 \end{bmatrix}$$

is orthogonally similar to a diagonal matrix. You are given that the eigenvalues are $\lambda_1 = 27$ and $\lambda_2 = \lambda_3 = -9$. Moreover, the eigenvectors corresponding to $\lambda_1 = 27$ are of the form $\alpha \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ while the eigenvectors for $\lambda_2 = \lambda_3 = -9$ are of the form $\beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Problem #7: Let $A = \begin{pmatrix} 4 & 1 & -2 \\ 1 & 4 & 2 \\ -2 & 2 & 1 \end{pmatrix}$. The characteristic polynomial of A is $-(\lambda+1)(\lambda-5)^2$.

(a) What property does A have that guarantees it is unitarily diagonalizable?

(b) Find a unitary matrix U and a diagonal matrix D so that $D = U^{-1}AU$.

Problem #8: The matrix $B = \begin{pmatrix} 1 & 4 \\ -2 & 7 \end{pmatrix}$ has eigenvectors $(1,1)$ and $(2,1)$.

(a) Find the eigenvalues of B.

(b) Solve the initial value problem $\begin{cases} \dfrac{dy_1}{dt} = y_1 + 4y_2 \\ \dfrac{dy_2}{dt} = -2y_1 + 7y_2 \end{cases}$ and $\begin{cases} y_1(0) = 1 \\ y_2(0) = 2 \end{cases}$.

Problem #9: The matrix $D = \begin{pmatrix} 9 & 1 & -2 & -6 \\ -19 & -3 & 4 & 11 \\ 8 & 1 & -3 & -4 \\ 11 & 1 & -2 & -8 \end{pmatrix}$ has eigenvalues -1 and -2.

(a) What is the characteristic polynomial of D?

(b) What are the algebraic and geometric multiplicities of $\lambda = -1$ and $\lambda = -2$?

$\lambda = -1$; algebraic ______________ geometric ______________.

$\lambda = -2$; algebraic ______________ geometric ______________.

(c) Solve the initial value problem: $\begin{cases} \dfrac{dY}{dt} = DY \\ Y(0) = C \end{cases}$ where $C = (3,2,0,6)$.

Problem #10: Read the paragraphs on the following page, then answer the following questions:

Let $P = \begin{pmatrix} -2 & 1 & -1 & 0 \\ 1 & -3 & 0 & 1 \\ -1 & 0 & -3 & 3 \\ 0 & 1 & 3 & -4 \end{pmatrix}$ Either find a non-zero vector v so that $\langle P\mathbf{v}, \mathbf{v}\rangle = 0$ or explain why there is no such vector.

Let $Q = \begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 4 & 1 \\ -1 & -1 & 1 & 3 \end{pmatrix}$ Either find a non-zero vector w so that $\langle Q\mathbf{w}, \mathbf{w}\rangle = 0$ or explain why there is no such vector.

Let $R = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 3 & 3 & 0 \\ -1 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ Either find a non-zero vector x so that $\langle R\mathbf{x}, \mathbf{x}\rangle = 0$ or explain why there is no such vector.

Reading for Problem #10: Let A be an $n \times n$ Hermitian matrix. We wish to investigate the equality $\langle A\mathbf{v}, \mathbf{v}\rangle = 0$ for non-zero vectors $\mathbf{v}$.

Notice, first of all, that if A is not invertible, there is a non-zero vector $\mathbf{v}$ so that $A\mathbf{v} = \mathbf{0}$. It follows trivially for this vector that

$$\langle A\mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{0}, \mathbf{v}\rangle = 0.$$

If A is invertible, then there is an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ for $\mathbf{C}^n$ consisting of eigenvectors for A, say $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$. The eigenvalues of A are real numbers and we assume that they have been arranged so that

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

Since A is invertible, none of the eigenvalues is 0. If all of the eigenvalues are positive, that is, if $\lambda_1 > 0$, then for $\mathbf{v}$ in $\mathbf{C}^n$ with $\mathbf{v} \neq \mathbf{0}$, there are scalars α_j so that

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n$$

and not all of the scalars are zero. Now

$$\begin{aligned}
\langle A\mathbf{v}, \mathbf{v}\rangle &= \langle A\left(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n\right), \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n\rangle \\
&= \langle \alpha_1\lambda_1\mathbf{u}_1 + \alpha_2\lambda_2\mathbf{u}_2 + \cdots + \alpha_n\lambda_k\mathbf{u}_n, \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n\rangle \\
&= \lambda_1|\alpha_1|^2 + \lambda_2|\alpha_2|^2 + \cdots + \lambda_n|\alpha_n|^2.
\end{aligned}$$

But since $\lambda_j \ge \lambda_1$, this shows

$$\langle A\mathbf{y}, \mathbf{y}\rangle \ge \lambda_1|\alpha_1|^2 + \lambda_1|\alpha_2|^2 + \cdots + \lambda_1|\alpha_n|^2 > 0$$

since $\lambda_1 > 0$ and at least one of the α_j is non-zero. In other words, in this case, $\langle A\mathbf{v}, \mathbf{v}\rangle > 0$ for all non-zero $\mathbf{v}$.

Similarly, if all the eigenvalues of A are negative, then $\lambda_n < 0$ and for $\mathbf{v}$ as above,

$$\begin{aligned}
\langle A\mathbf{v}, \mathbf{v}\rangle &= \lambda_1|\alpha_1|^2 + \lambda_2|\alpha_2|^2 + \cdots + \lambda_n|\alpha_n|^2 \\
&\le \lambda_n|\alpha_1|^2 + \lambda_n|\alpha_2|^2 + \cdots + \lambda_n|\alpha_n|^2 < 0
\end{aligned}$$

since $\lambda_n < 0$ and at least one of the α_j is non-zero. Thus, in this case, $\langle A\mathbf{v}, \mathbf{v}\rangle < 0$ for all non-zero $\mathbf{v}$.

However, if $\lambda_1 < 0$ and $\lambda_n > 0$, then let $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_n\mathbf{u}_n$ where $\alpha_1 = \sqrt{\lambda_n}$ and $\alpha_n = \sqrt{|\lambda_1|}$. Then $\alpha_1^2 = \lambda_n$ and $\alpha_n^2 = |\lambda_1| = -\lambda_1$ so

$$\langle A\mathbf{v}, \mathbf{v}\rangle = \lambda_1\alpha_1^2 + \lambda_n|\alpha_n|^2 = \lambda_1\lambda_n - \lambda_n\lambda_1 = 0.$$

Problem #11: The characteristic polynomial of the following hermitian matrix

$$A = \begin{bmatrix} 6 & 3-i & -2i \\ 3+i & 5 & -2+2i \\ 2i & -2-2i & 5 \end{bmatrix}$$

is

$$p_A(\lambda) = \lambda(7-\lambda)(9-\lambda).$$

Find a basis of eigenvectors and verify that they are orthogonal.

Problem #12: **Fibonacci's rabbits, XIII century:** Suppose that newly born pairs of rabbits produce no offspring during the first month of their lives, but each pair produces one new pair each subsequent month. Starting with $F_1 = 1$ newly born pair in the first month, find the number F_k of pairs in the kth month, assuming that no rabbit dies.

In the kth month, the number of pairs of rabbits is

$$\begin{aligned} F_k &= \text{Number of pairs alive the preceeding month } + \\ &\quad + \text{ Number of newly born pairs for the } k\text{th month.} \end{aligned}$$

Since our rabbits do not produce offspring during the first month of their lives, we see that the number of newly born pairs for the kth month is the number F_{k-2} of pairs alive two months before. Thus we can write the equation above as

$$\boxed{F_k = F_{k-1} + F_{k-2}.} \tag{1}$$

This is called the *Fibonacci relation*. It is convenient to set $F_0 = 0$, denoting 0 pairs for month 0 before the arrival of the first newly born pair, which is presumably a gift. Thus the sequence

$$F_0, F_1, F_2, F_3, \ldots, F_k, \ldots$$

for the pairs of rabbits becomes the *Fibonacci sequence*

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots,$$

where each term starting with $F_2 = 0 + 1 = 1$ is the sum of the two preceding terms.

The Fibonacci sequence occurs naturally in a surprising number of places. For example, leaves appear in a spiral pattern along a branch. Some trees have 5 growths of leaves for every 2 turns, others have 8 growths for every 3 turns, and still others have 13 growths for every 5 turns.

The kth Fibonacci number can be found by simply writing out enough terms of the sequence: that can be a tedious task, even if we want to compute only F_{30}. Linear Algebra gives us another approach to this problem. The Fibonacci relation (1) can be expressed in matrix form. We see that

$$\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix}.$$

Thus if we set

$$\mathbf{x}_k = \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

we find that

$$\mathbf{x}_k = A\mathbf{x}_{k-1}. \tag{2}$$

Applying (2) repeatedly, we see that $\mathbf{x}_2 = A\mathbf{x}_1$, $\mathbf{x}_3 = A\mathbf{x}_2 = A^2\mathbf{x}_1$, $\mathbf{x}_4 = A\mathbf{x}_3 = A^3\mathbf{x}_1$, and in general

$$\mathbf{x}_k = A^{k-1}\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus we can compute the kth Fibonacci number F_k by finding A^{k-1} and multiplying it on the right by the colum vector $\mathbf{x}_1$.

The characteristic equation of A is

$$\lambda^2 - \lambda - 1 = 0.$$

It is easy to chech that A has the following distinct eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

and that we can choose the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ \sqrt{5}-1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ \sqrt{5}+1 \end{bmatrix}.$$

as corresponding eigenvectors. Thus if we take

$$S = \begin{bmatrix} 2 & -2 \\ \sqrt{5}-1 & \sqrt{5}+1 \end{bmatrix}$$

we conclude that A can be diagonalized, i.e.,

$$S^{-1}AS = D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

Therefore

$$\begin{aligned} \mathbf{x}_{k+1} &= \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k\mathbf{x}_1 = SD^kS^{-1} = \\ &= \underbrace{\left(\frac{\sqrt{5}+1}{4\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k \begin{bmatrix} 2 \\ \sqrt{5}-1 \end{bmatrix}} - \left(\frac{\sqrt{5}-1}{4\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k \begin{bmatrix} -2 \\ \sqrt{5}+1 \end{bmatrix}. \end{aligned}$$

For large k, the kth power of the eigenvalue λ_1 dominates so that $A^k\mathbf{x}_1$ is approximately equal to the underbraced portion of the previous formula. Computing the second component of $A^k\mathbf{x}_1$ we find that

$$F_k \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k \qquad \text{for } k \text{ large}.$$

This show that F_k increases exponentially with k ... as expected for a population of rabbits.

k	5	10	20
$F_k =$	5	55	6,765
$F_k \approx$	4.9597	55.0036	6,765

Problem #13 Competing populations: a sad story! Mathematical models often lead to an $n \times 1$ matrix $\mathbf{x}_i$ that describes the state of some complicated system at time i and to an $n \times n$ matrix A that represents the internal workings of the system in such a way that

$$\mathbf{x}_{i+1} = A\mathbf{x}_i$$

models the evolution of the system overtime. Denoting the initial state by $\mathbf{x}_0$ allows us to write this equivalently as

$$\mathbf{x}_i = A^i\mathbf{x}_0 \qquad \text{for } i \geq 0.$$

Let us consider the situation in which two populations compete with one another. We denote the numbers in these populations by f_i and c_i, which you may view as counting *foxes* and *chickens*.

- Assume that the chickens, without any raiding foxes to harrass them, would have a birth rate exceeding the death rate; to be specific, imagine that we would have $c_{i+1} = 1.2c_i$.
- Without chickens to feed upon, we would expect the foxes to die out: say, $f_{i+1} = 0.6f_i$.

We want to model what happens when the foxes succed in eating some number of chickens each time of period, and we assume that this would allow an increase in the fox population proportional to that number of chickens killed: e.g., $f_{i+1} = 0.6f_i + 0.5c_i$. The chicken population would of course decrease because of the raiding foxes, so we take $c_{i+1} = 1.2c_i - kf_i$, where k reflects the kill rate of chicken by foxes. Supposing that $c_0 = 1000$ and $f_0 = 100$, we obtain the following model

$$\begin{bmatrix} f_{i+1} \\ c_{i+1} \end{bmatrix} = \mathbf{x}_{i+1} = A\mathbf{x}_i = \begin{bmatrix} 0.6 & 0.5 \\ -k & 1.2 \end{bmatrix} \begin{bmatrix} f_i \\ c_i \end{bmatrix} \qquad \mathbf{x}_0 = \begin{bmatrix} 100 \\ 1000 \end{bmatrix}.$$

What happens as time passes? Does the system tend to equilibrium? Do the states $\mathbf{x}_i$ become arbitrarily large? Do the states tend to zero? Do they oscillate?
Choose for example $k = 0.18$ and show that whenever $i \longrightarrow \infty$ then A^i becomes the zero matrix. Hence the configuaration of our population tends to zero. That is, the choice $k = 0.18$ causes the chicken population to be killed off, which in turn leads to the death (isn't it called 'starvation'?) of the foxes, no matter what the starting distribution between foxes and chickens. (Unfortunately, in this case the matrix A is not diagonalizable. What can you do?)

Problem #14: Find a real number a such that the 3×3 matrix

$$U = \begin{bmatrix} a & \frac{1}{2}i & \frac{1}{2}a(2i-1) \\ ia & \frac{1}{2}(1+i) & \frac{1}{2}a(1-i) \\ a & -\frac{1}{2} & \frac{1}{2}a(2-i) \end{bmatrix}$$

is unitary.