Practice Test
Chapters 4, 6, 8

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Linear Algebra for Engineering and Science

(Mathematics 511 – Summer 1997)

Exercise #1: Let V be the vector space of all 2×2 matrices over $\mathbf{R}$.
Prove that V has dimension 4 by exhibiting a basis for V which has four elements.
Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix} \qquad \text{for all } x,y,z \in \mathbf{R}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix} \qquad \text{for all } a,b,c \in \mathbf{R}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Find the dimensions of W_1, W_2, W_1+W_2, and $W_1 \cap W_2$; check that the formula

$$\dim W_1 + \dim W_2 = \dim(W_1+W_2) + \dim(W_1 \cap W_2)$$

holds.

Problem #2: Let $\varphi(t)$ be a non-negative continuous real-valued function on the closed interval $[0,1]$ which is positive on $(0,1)$. For any pair of continuous real-valued functions f and g on $[0,1]$, define

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\varphi(t)dt.$$

(a) Show that $\langle f,g\rangle$ is an inner product on the vector space of all real-valued continuous functions on $[0,1]$.

(b) If $\varphi(t) = 2t$ for all $t \in [0,1]$, find the norm of $\left(t^2+1\right)^{-1}$ with respect to this inner product.

(c) If $\varphi(t) = 2t$ for all $t \in [0,1]$, find an orthogonal basis for the subspace P_2 of all polynomials of degree ≤ 2. Check that your polynomials are indeed orthogonal.

Exercise #3: If V is an inner product space, the cosine of the angle between two vectors $\mathbf{v},\mathbf{w} \in V$ is defined by

$$\cos(\theta) \stackrel{\text{def}}{=} \frac{\langle \mathbf{v},\mathbf{w}\rangle}{\| \mathbf{v} \| \| \mathbf{w} \|}.$$

Note that the Cauchy-Schwarz inequality implies that $|\cos(\theta)| \leq 1$, as in the usual case.
Let V be the real vector space of all continuous real valued functions on $[0,1]$. Consider the inner product on V defined, for f and g in V, by

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Find the angle between $f(t) = t$ and $g(t) = t^2 - t + 1$.

Exercise #4: Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

in $\mathbf{R}^3$ equipped with the standard inner product.

(a) Apply the Gram-Schmidt orthonormalization process to the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ in order to obtain an orthonormal basis for $\mathbf{R}^3$.

(b) Write the QR decomposition for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Exercise #5: Let V be an inner product space and let W be one of its subspaces. If P_W denotes the orthogonal projection onto the subspace W, show that for every $\mathbf{v} \in V$

$$\| \mathbf{v} \|^2 = \| P_W(\mathbf{v}) \|^2 + \| \mathbf{v} - P_W(\mathbf{v}) \|^2 .$$

Problem #6: Give $\mathbf{R}^3$ the standard inner product.

(a) Write the matrix P_W that projects every point in $\mathbf{R}^3$ onto the subspace $W = \{(x,y,z) \in \mathbf{R}^3 \mid x+y-z=0\}$.

(b) Write the matrix $P_{W^\perp}$ that projects every point in $\mathbf{R}^3$ onto the subspace $W^\perp$.

Exercise #7: Let V be the real vector space of all 2×2 matrices over $\mathbf{R}$. We define the trace of a matrix A to be the sum of its elements on the main diagonal, i.e.

$$\text{trace}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

(a) Verify that for any $A, B \in V$ the map defined by $\langle A, B \rangle_{\text{tr}} = \text{trace}(AB^{\text{T}})$, i.e.

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right\rangle_{\text{tr}} = \text{trace}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix}\right) = ax + by + cz + dw,$$

gives an inner product on V.

(b) Let W_1 be the two dimensional subspace of V consisting of all 2×2 matrices of the form

$$\begin{bmatrix} a & -b \\ -a & b \end{bmatrix}.$$

Find (using Gram-Schmidt orthonormalization process) a basis for W_1 with respect to the inner product $\langle -, - \rangle_{\text{tr}}$.

(c) Why do you think that $\langle A, B \rangle_{\text{tr}} = \text{trace}(AB)$ does not work?

Problem #8: Let M be the subspace of $\mathbf{R}^5$ spanned by $\mathbf{u}_1 = (1,1,-1,-1,0)$, $\mathbf{u}_2 = (1,-1,-1,0,1)$, $\mathbf{u}_3 = (-1,-1,0,1,1)$, and $\mathbf{u}_4 = (1,-1,-2,0,2)$. Find an orthonormal basis for $M^{\perp}$.

Problem #9: Let W be the subspace of $\mathbf{R}^5$ spanned by $\mathbf{w}_1 = (1,1,0,-1,1)$, $\mathbf{w}_2 = (0,2,-1,1,1)$, and $\mathbf{w}_3 = (-1,1,1,-2,1)$. Find $\mathbf{x}$ in W and $\mathbf{y}$ in $W^{\perp}$ so that $\mathbf{x}+\mathbf{y} = (1,1,-1,-1,1)$.

Problem #10: A simple model for the air resistance of a moving body is $R = av + bv^2$ where R is the force due to air resistance, v is the velocity of the object, and a and b are constants that depend on the shape of the object. Find the best (least squares) estimates for a and b based on the data below:

Air Resistance Data

Force	Velocity
2	1
7	2
18	5
45	8

Problem #11: (a) What is a least squares solution $\mathbf{X_0}$ of a system $A\mathbf{X} = \mathbf{b}$? Why is it the *best solution*?

(b) Solve $A\mathbf{X} = \mathbf{b}$ by least squares and find $\mathbf{b_0} = A\mathbf{X_0}$ if

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

Verify that the error $\mathbf{b} - \mathbf{b_0}$ is orthogonal to the columns of A.

Problem #12: (a) Let B be the matrix

$$B = \begin{pmatrix} 5 & 0 & -2 & -2 \\ -3 & -2 & 8 & 18 \\ 3 & 3 & -3 & -12 \\ -2 & -3 & 5 & 14 \end{pmatrix}$$

Show that $\mathbf{x} = (1,1,1,0)$ and $\mathbf{y} = (2,-3,3,-2)$ are eigenvectors of B, but $\mathbf{z} = (1,1,0,0)$ is not.

(b) Find the eigenvector of B with eigenvalue 3 that is closest to $\mathbf{z}$.

Exercise #13: (a) Suppose $\mathbf{u}$ is an eigenvector for A' and $\mathbf{v}$ is orthogonal to $\mathbf{u}$. Show that $A\mathbf{v}$ is also orthogonal to $\mathbf{u}$.

(b) Use part (a) (whether you proved it or not) to show that if A is a 2×2 Hermitian matrix, and $\mathbf{u}$ is an eigenvector of A, then any non-zero vector $\mathbf{v}$ that is orthogonal to $\mathbf{u}$ is also an eigenvector of A.

Exercise #14: Let A and B be two invertible $n\times n$ matrices. Show that AB and BA have the same eigenvalues.

Exercise #15: Let A be an $n\times n$ Hermitian (self-adjoint) matrix. Show that the eigenvalues of A are real. Show that $A - iI_n$ is non-singular and that $U = (A - iI_n)^{-1}(A + iI_n)$ is a unitary matrix.

Exercise #16: **(From a recent qualifying test in Statistics).** Let A_1 and A_2 be two $n \times n$ diagonalizable matrices with distinct non-zero characteristic roots. Assume that

$$A_1 A_2 = A_2 A_1.$$

Show that A_1 and A_2 are simultaneously diagonalizable, i.e., there exists an invertible $n \times n$ matrix S such that

$$S^{-1} A_1 S \qquad \text{and} \qquad S^{-1} A_2 S$$

are diagonal matrices.

SOLUTION: We know from class, that a $n \times n$ matrix A is diagonalizable if and only if we can find a basis for $\mathbf{R}^n$ consisting of n distinct non-zero eigenvectors for A. By letting S to be the matrix whose columns are the distinct eigenvectors of A one has that $S^{-1}AS = D$, where D is the diagonal matrix with the eigenvalues of A on its main diagonal.
By assumption we know that both A_1 and A_2 have n distinct, non-zero eigenvalues. For each, eigenvalue we can find an eigenvector. By putting all of them together we find two matrices S_1 and S_2 such that $S_1^{-1} A_1 S_1$ and $S_2^{-1} A_2 S_2$ are diagonal matrices.
The goal of the problem is to show that we can find a single matrix S that will simultaneously diagonalize both A_1 and A_2.
In order to solve the problem, it will be enough to show that we can find n distinct vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ that are simultaneously eigenvectors for both A_1 and A_2. S will then be the matrix with $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as columns.
In general, let $\mathbf{v}$ be an eigenvector for A_1 associated to the eigenvalue $\lambda \neq 0$. It is enough to show that $\mathbf{v}$ is also an eigenvector for A_2.
By assumption, $A_1 \mathbf{v} = \lambda \mathbf{v}$ or, equivalently, $(A_1 - \lambda I_n)\mathbf{v} = \mathbf{0}$. Multiply both sides by A_2. We get $A_2(A_1 - \lambda I_n)\mathbf{v} = A_2 \mathbf{0} = \mathbf{0}$. Let us now use the property that A_1 and A_2 commute. We get that $A_2(A_1 - \lambda I_n)\mathbf{v} = (A_2 A_1 - \lambda A_2)\mathbf{v} = (A_1 A_2 - \lambda A_2)\mathbf{v} = (A_1 - \lambda I_n)(A_2 \mathbf{v}) = \mathbf{0}$. Therefore $A_2 \mathbf{v} \in \mathrm{Null}(A_1 - \lambda I_n)$.
We now use the property that the eigenvalues of A_i, for $i = 1, 2$, are all distinct. This hypothesis means that the eigenspace corresponding to each eigenvalue of A_i is one-dimensional. In particular, for our general eigenvector $\mathbf{v}$ we have that $\mathrm{Null}(A_1 - \lambda I_n) = \mathrm{span}\{\mathbf{v}\}$.
Finally, we have that $A_2 \mathbf{v} \in \mathrm{Null}(A_1 - \lambda I_n) = \mathrm{span}\{\mathbf{v}\}$. Thus $A_2 \mathbf{v} = \mu \mathbf{v}$ for some $\mu \in \mathbf{R}$. Hence the assertion.

COMMENT: It seems to me that the assumption that the eigenvalues are non-zero is not necessary in this problem.

The problems below are intended only for EXTRA CREDIT ... and YOUR MATHEMATICAL CULTURE.

Problem I: **Definition:** *Let A be an $m \times n$ matrix with entries in $\mathbf{R}$; any $n \times m$ matrix A^+ with entries in $\mathbf{R}$ that satisfies the following four properties*

1. $AA^+A = A$;
2. $A^+AA^+ = A^+$;
3. $AA^+ = (AA^+)^{\mathrm{T}}$;
4. $A^+A = (A^+A)^{\mathrm{T}}$

is said to be a <u>*Moore–Penrose pseudoinverse*</u> *of A.*

We list some of the properties of the Moore–Penrose pseudoinverse of A. Prove as many properties as possible.

P1. A^+ is uniquely determined by conditions *1., 2., 3.,* and *4.*

P2. If A is a non-singular $n \times n$ matrix then $A^+ = A^{-1}$.

P3. If $\operatorname{rank}(A) = n$, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

P4. If $\operatorname{rank}(A) = m$, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$.

P5. If A is a projection matrix, then $A^+ = A$.

P6.
- $\operatorname{rank}(A) = \operatorname{rank}(A^+)$;
- $(A^+)^+ = A$;
- $(A^+)^{\mathrm{T}} = (A^{\mathrm{T}})^+$;
- $(cA)^+ = c^{-1}A^+$ if $c \neq 0$;
- if $A = A^{\mathrm{T}}$ then $A^+ = (A^+)^{\mathrm{T}}$.

P7. For every matrix A the Moore–Penrose pseudoinverse A^+ always exists.

P8. A *least squares solution* to the linear equations $A\mathbf{X} = \mathbf{b}$ is a vector $\mathbf{X}$ such that $\| \mathbf{X} \|$ is minimized among all vectors $\mathbf{X}$ for which $\| A\mathbf{X} - \mathbf{b} \|$ is minimal. Show that $\mathbf{X} = A^+\mathbf{b}$ is a least squares solution to $A\mathbf{X} = \mathbf{b}$.

Problem II: **Definition:** *Let T be a linear transformation on an inner product space V. Then we say that T has an adjoint on V if there exists a linear operator T' on V such that*

$$\langle T(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T'(\mathbf{w}) \rangle$$

for all $\mathbf{v}$ and $\mathbf{w}$ in V.

Remark: It is a fact that every linear transformation on a finite dimensional inner product space V has an adjoint on V. In the infinite dimensional case this is not always true. But in any case there is at most one such operator T'; when it exists, we call it <u>the</u> *adjoint* of T.

(*a*) How can you view Problem #2 in terms of adjoints?

(*b*) Let $V = M_{n\times n}(\mathbf{R})$ with the inner product $\langle A, B \rangle = \operatorname{trace}(AB^{\mathrm{T}})$. Let M be a fixed $n \times n$ matrix over $\mathbf{R}$ and let L_M be the operator 'left multiplication' by M, i.e. $L_M(A) = MA$. Show that the adjoint $L_M{}'$ of L_M is the operator 'left multiplication' by M^{T}, i.e. $L_M{}' = L_{M^{\mathrm{T}}}$.

Exercise III: **Definition:** *Let W be a subspace of a complex (real) inner product space V. A linear transformation $T\colon W \longrightarrow V$ is said to be hermitian (self-adjoint) if for all $\mathbf{v}, \mathbf{w}$ in W*

$$\langle T(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle.$$

Remark: *In the case in which $W = V = \mathbf{C}^n$ with the standard inner product and the linear transformation T is represented by a matrix A, the previous condition reads as follows: $(A\mathbf{v})'\mathbf{w} = \mathbf{v}'(A\mathbf{w})$, i.e., $A = A'$.*

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Let $V = C[a,b]$ be the real vector space of all continuos real valued functions defined on the closed interval $[a,b]$ with the real inner product defined by

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt$$

for every pair of functions f, g in V.
Let W be the subspace of V consisting of all functions f with continuous second derivative on $[a,b]$ which satisfy the boundary condition

$$p(a)f(a) = p(b)f(b) = 0,$$

where p is a fixed function in V with continuous second derivative. Let q be another fixed function in V and define the operator $T\colon W \longrightarrow V = C[a,b]$ by

$$T(f) = (pf')' + qf.$$

T is the so-called *Sturm-Liouville operator*.

(a) Show that T is a symmetric operator.

(b) Let f and g be two functions in W such that $T(f) = 0 = T(g)$ and suppose that $p(t) > 0$. Show that the Wronskian of f and g is given by $C/p(t)$, where C is a constant.

(c) A nonzero function f in W is an eigenvector of T if there exists a $\lambda \in \mathbf{R}$ such that $T(f) = \lambda f$. Show that if f and g are two eigenvectors of T corresponding to distinct eigenvalues, then $\langle f, g \rangle = 0$.

(d) The *Legendre polynomials* are defined by the equation

$$P_n(t) = \frac{1}{2^n n!} f_n^{(n)}(t),$$

where $f_n(t) = (t^2-1)^n$.
Following the next steps, show that $P_n(t)$ is an eigenvector associated with the eigenvalue $n(n+1)$ of the Sturm-Liouville operator T on the interval $[-1,1]$ given by $T(f) = (pf')'$, where $p(t) = t^2 - 1$.

- Show that $(t^2-1)f_n'(t) = 2nt f_n(t)$;
- Differentiate both sides of the previous equation in order to get

$$(t^2-1)f_n^{(n+2)}(t) + 2t(n+1)f_n^{(n+1)}(t) + n(n+1)f_n^{(n)}(t) =$$

$$= 2nt f_n^{(n+1)}(t) + 2n(n+1)f_n^{(n)}(t);$$

- Recognize that the above equation can be written as

$$\big((t^2-1)P_n'(t)\big)' = n(n+1)P_n(t).$$