(12) 1. Solve $Ax = b$ by solving the triangular systems $Lc = b$ and $Ux = c$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Show the intermediate step.

What is the rank of A? _____

(8) 2. (a) How many 2 by 2 matrices are there with each entry either 0 or 1?

(b) How many of the matrices in part(a) are invertible?

(8) 3. (a) How many 3 by 3 permutation matrices are there?

(b) What is the smallest positive integer k such that all 3 by 3 permutation matrices P satisfy $P^k = I$?

(18) 4. For each of the situations below, decide which of the statements below are true about systems with this matrix as coefficient matrix. Include all correct responses.

(a) A is a 7 by 12 matrix.

If the rank of A is 5, the system $Ax = b$ _____

If the rank of A is 7, the system $Ax = b$ _____

If the rank of A is 9, the system $Ax = b$ _____

(b) C is a 11 by 7 matrix

If the rank of C is 5, the system $Cx = b$ _____

If the rank of C is 7, the system $Cx = b$ _____

If the rank of C is 9, the system $Cx = b$ _____

(i) has at least one solution for every b

(ii) has no solutions for some b

(iii) has at most one solution for every b

(iv) has infinitely many solutions for some b

(v) has infinitely many solutions for every b

(vi) The given information is contradictory, no such system is possible.

(8) 5. Find a basis for the subspace V of all vectors in $\mathbb{R}^4$ whose first three components are equal. What is the dimension of V?

(10) 6. Write down a 3 by 3 matrix A which transforms the coordinate vectors e_1, e_2, e_3 into the three vectors

$$v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Is the matrix A invertible?

(10) 7. Assume that A is a 4 by 4 matrix and that the system $Ax = b$, where $b = \begin{bmatrix} 8 \\ 1 \\ 4 \\ 5 \end{bmatrix}$ has solutions $v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$.

(a) Write down a nontrivial solution of the homogeneous system $Ax = 0$.

(b) Write down a solution to the system $Ax = b$ other than v_1 and v_2.

(10) 8. What is the dimension of the vector space V of 2 by 2 symmetric matrices? Write down a basis for V.

(16) 9. State true or false and briefly justify.

(i) If A is an n by n matrix, then the row space of A equals the column space of A.

(ii) If v is a nonzero vector in $\mathbb{R}^4$ and w is a nonzero vector in $\mathbb{R}^5$, then there exists a 4 by 5 matrix A whose column space is spanned by v and whose row space is spanned by w.

(iii) If V is a vector space of dimension 7 and W is a subspace of V of dimension 5, then every basis for V can be reduced to a basis for W by removing two vectors.

(iv) If the columns of A are linearly independent, then $Ax = b$ has at most one solution for every b.