Final
Chapters 1, 2, 3, 4, 5, and 6

Date: August 3, 1995
Time: 3 hours

Linear Algebra and Applications

(Mathematics 642:550:01 – Summer 1995)

Problem #1: (15pts) Give a complete analysis of the linear transformation $T\colon \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -4 & -4 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}$$

(a) Find a basis for the nullspace $\mathcal{N}$;

(b) For what explicit 3×1 column vectors $Y = (a, b, c)^{\mathrm{T}}$ does the equation $AX = Y$ have solutions X?

(c) Write a decomposition $A = LU$.

Problem #2: (10pts) Consider the set of vectors $(x_1, x_2, x_3, x_4)^{\mathrm{T}}$ in $\mathbf{R}^4$ which satisfy the condition

$$x_1 - x_2 + x_3 - x_4 = 0.$$

(a) Verify that this collection of vectors forms a subspace of $\mathbf{R}^4$;

(b) Find a basis for this subspace;

(c) Observe that $\mathbf{u} = (1, 2, 3, 2)^{\mathrm{T}}$ and $\mathbf{v} = (1, 1, 1, 1)^{\mathrm{T}}$ belong to this subspace; find a basis for the subspace which includes $\mathbf{u}$ and $\mathbf{v}$;

(d) Extend the basis in (c) to a basis for the whole $\mathbf{R}^4$;

(e) Find an orthonormal vector to the subspace.

Problem #3: (10pts) Suppose that $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ form a linearly independent set. Define the new vectors

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \\ \mathbf{v}_2 &= \mathbf{u}_1 + \alpha \mathbf{u}_2 \\ \mathbf{v}_3 &= \mathbf{u}_2 + \beta \mathbf{u}_3. \end{aligned}$$

Find a general conditions on α and β in order that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ form a linearly independent set.

Problem #4: (10pts) Show that two matrices A and B are similar if and only if they have the same Jordan canonical form.

Problem #5: (15pts) Verify that the matrix

$$A = \begin{bmatrix} 7 & -16 & -8 \\ -16 & 7 & 8 \\ -8 & 8 & -5 \end{bmatrix}$$

is orthogonally similar to a diagonal matrix. You are given that the eigenvalues are

$$\lambda_1 = 27 \quad \text{and} \quad \lambda_2 = \lambda_3 = -9.$$

Moreover, the eigenvectors corresponding to $\lambda_1 = 27$ are of the form

$$\alpha \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix};$$

the eigenvectors for $\lambda_2 = \lambda_3 = -9$ are of the form

$$\beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Problem #6: (15pts) The characteristic polynomial of the following hermitian matrix

$$A = \begin{bmatrix} 6 & 3-i & -2i \\ 3+i & 5 & -2+2i \\ 2i & -2-2i & 5 \end{bmatrix}$$

is

$$p_A(\lambda) = \lambda(7-\lambda)(9-\lambda).$$

Find a basis of eigenvectors and verify that they are orthogonal.

Problem #7: (15pts) Use the methods of linear algebra to solve the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Problem #8: (10pts) Consider the quadratic form

$$Q(x) = Q(x_1, x_2, x_3) = 2x_1^2 + 4x_1x_3 + x_2^2 - x_3^2.$$

(a) Write a symmetric matrix A such that $Q(x) = \langle x, Ax \rangle = x^{\mathrm{T}} Ax$.

(b) Is $Q(x)$ positive, or negative (semi)definite, or indefinite?