Homework Assignment # 2 Due Date: July 13
Chapter 2: Vector Spaces and Linear Equations

Linear Algebra and Applications

(Mathematics 642:550:01 – Summer 1995)

Exercise #1: Decide the dependence or independence of:

(a) $v_1 - v_2, v_2 - v_3, v_3 - v_4$, and $v_4 - v_1$ for any vectors v_1, v_2, v_3, and v_4;

(b) $(1, 1, 2), (1, 2, 1)$, and $(3, 1, 1)$.

Exercise #2: In the vector space V of all cubic polynomials $P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, let S be the subset of polynomials with

$$\int_0^1 P(x)\, dx = 0.$$

Verify that S is a subspace of V and find a basis.

Exercise #3: Let V be the vector space of all 2×2 matrices over a field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.
Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Find the dimensions of $W_1, W_2, W_1 + W_2$, and $W_1 \cap W_2$; check that the formula

$$\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

holds.

Exercise #4: Let P be a fixed $m \times m$ matrix with entries in the field F and let Q be a fixed $n \times n$ matrix over F. Define a function T from $M_{m \times n}(F)$ into itself by

$$T(A) = PAQ.$$

Verify that T is a linear transformation.

Exercise #5: Let T be the linear transformation from $\mathbf{R}^2$ into $\mathbf{R}^2$ defined by

$$T(x_1, x_2) = (-x_2, x_1).$$

(a) What is the matrix A of T in the standard ordered basis $\{e_1, e_2\}$ for $\mathbf{R}^2$?

(b) What is the matrix A' of T in the ordered basis $\{e_1' = e_1 + 2e_2, e_2' = e_1 - e_2\}$?

(c) Verify that $A' = P^{-1}AP$, where P is the matrix

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Exercise #6: What is the echelon form U of

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{bmatrix}?$$

What are the dimensions of its four fundamental subspaces?

Exercise #7: Let T be the linear function from $\mathbf{R}^3$ into $\mathbf{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

(a) Write the matrix A of T in the standard ordered basis for $\mathbf{R}^3$.

(b) If (a, b, c) is a vector in $\mathbf{R}^3$, what are the conditions on a, b, and c that the vector be in the range of T? What is the rank of T?

(c) What are the conditions on a, b, and c that the vector (a, b, c) be in the nullspace of T? What is the nullity of T?

(d) Does your answer in (b) and (c) confirm the formula

$$\text{rank}(T) + \text{nullity}(T) = \dim(\mathbf{R}^3)?$$