Homework Assignment # 3 Due Date: July 20
Chapter 3: Ortogonality

Linear Algebra and Applications

(Mathematics 642:550:01 – Summer 1995)

Exercise #1: If V is an inner product space, the cosine of the angle between two vectors $v, w \in V$ is defined by

$$\cos(\theta) \stackrel{\text{def}}{=} \frac{\langle v, w \rangle}{\| v \| \| w \|}.$$

Note that the Cauchy-Schwarz inequality implies that $|\cos(\theta)| \leq 1$, as in the usual case. Let V be the real vector space of all continuous real valued functions on $[0, 1]$. Consider the inner product on V defined, for f and g in V, by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt.$$

Find the angle between $f(t) = t$ and $g(t) = t^2 - t + 1$.

Exercise #2: Consider the vectors

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

in $\mathbf{R}^3$ equipped with the standard inner product.

(a) Apply the Gram-Schmidt orthonormalization process to the vectors v_1, v_2, and v_3 in order to obtain an orthonormal basis for $\mathbf{R}^3$.

(b) Write the QR decomposition for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Exercise #3: Give $\mathbf{R}^3$ the standard inner product.

(a) Write the matrix P_W that projects every point in $\mathbf{R}^3$ onto the subspace $W = \{(x, y, z) \in \mathbf{R}^3 \mid x + y + z = 0 \text{ and } x - z = 0\}$.

(b) Write (by direct calculation) the matrix $P_{W^\perp}$ that projects every point in $\mathbf{R}^3$ onto the subspace $W^\perp$. Verify that $P_{W^\perp} = I_3 - P_W$.

Exercise #4: Let V be an inner product space and let W be one of its subspaces. If P_W denotes the orthogonal projection onto the subspace W, show that for every $v \in V$

$$\| v \|^2 = \| P_W(v) \|^2 + \| v - P_W(v) \|^2 .$$

Exercise #5: Let V be the real vector space of all 2×2 matrices over $\mathbf{R}$. We define the trace of a matrix A to be the sum of its elements on the main diagonal, i.e.

$$\mathtt{trace}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

(a) Verify that for any $A, B \in V$ the map defined by $\langle A, B\rangle_{\mathrm{tr}} = \mathtt{trace}(AB^{\mathrm{T}})$, i.e.

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right\rangle_{\mathrm{tr}} = \mathtt{trace}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix}\right) = ax+by+cz+dw,$$

gives an inner product on V.

(b) Let W_1 be the two dimensional subspace of V consisting of all 2×2 matrices of the form

$$\begin{bmatrix} a & -b \\ -a & b \end{bmatrix}.$$

Find (using Gram-Schmidt orthonormalization process) a basis for W_1 with respect to the inner product $\langle -, -\rangle_{\mathrm{tr}}$.

(c) Why do you think that $\langle A, B\rangle_{\mathrm{tr}} = \mathtt{trace}(AB)$ does not work?

Exercise #6: Approximate (using the method you prefer) $y = x^4$ by a degree 2 polynomial between $x = 0$ and $x = 1$.

Exercise #7: Solve $Ax = b$ by least squares and find $p = A\overline{x}$ if

$$A = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix} \qquad b = \begin{bmatrix} -3 \\ 7 \\ 1 \\ 0 \\ 8 \end{bmatrix}.$$

Verify that the error $b - p$ is orthogonal to the columns of A.