Homework Assignment # 5 Due Date: August 1
Chapter 5: Eigenvalues and Eigenvectors

Linear Algebra and Applications

(Mathematics 642:550:01 – Summer 1995)

Exercise #1: Let A and B be two invertible $n \times n$ matrices. Show that AB and BA have the same eigenvalues.

Exercise #2: Diagonalize the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

What is the minimal polynomial of A?

Exercise #3: Find a real number a such that the 3×3 matrix

$$U = \begin{bmatrix} a & \frac{1}{2}i & \frac{1}{2}a(2i-1) \\ ia & \frac{1}{2}(1+i) & \frac{1}{2}a(1-i) \\ a & -\frac{1}{2} & \frac{1}{2}a(2-i) \end{bmatrix}$$

is unitary.

Exercise #4: Reduce to Jordan form the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Exercise #5: Find the minimal polynomial of the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Exercise #6: Solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ -1 & 25 & 8 \end{bmatrix}\mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise #7: Let A be an $n \times n$ hermitian matrix. Show that $A - iI_n$ is non-singular and that

$$(A - iI_n)^{-1}(A + iI_n)$$

is a unitary matrix.

Exercise #8: **OPTIONAL**

Definition *Let W be a subspace of a complex (real) inner product space V. A linear transformation $T: W \longrightarrow V$ is said to be hermitian (symmetric) if*

$$\langle T(x), y \rangle = \langle x, T(y) \rangle$$

for all x, y in W.

Remark *In the case in which $W = V = \mathbf{C}^n$ with the standard inner product and the linear transformation T is represented by a matrix A, the previous condition reads as follows*

$$(Ax)^H y = x^H (Ay),$$

i.e. $A = A^H$.

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Let $V = C(a, b)$ be the real vector space of all continuos real valued functions defined on the closed interval $[a, b]$ with the real inner product defined by

$$\langle f, g \rangle = \int_a^b f(t) g(t) dt$$

for every pair of functions f, g in V.
Let W be the subspace of V consisting of all functions f with continuous second derivative on $[a, b]$ which satisfy the boundary condition

$$p(a) f(a) = p(b) f(b) = 0,$$

where p is a fixed function in V with continuous second derivative. Let q be another fixed function in V and define the operator $T: W \longrightarrow V = C(a, b)$ by

$$T(f) = (pf')' + qf.$$

T is the so-called *Sturm-Liouville operator*.

(a) Show that T is a symmetric operator.

(b) Let f and g be two functions in W such that $T(f) = 0 = T(g)$ and suppose that $p(t) > 0$. Show that the Wronskian of f and g is given by $C/p(t)$, where C is a constant.

(c) A nonzero function f in W is an eigenvector of T if there exists a $\lambda \in \mathbf{R}$ such that $T(f) = \lambda f$. Show that if f and g are two eigenvectors of T corresponding to distinct eigenvalues, then $\langle f, g \rangle = 0$.

(d) The *Legendre polynomials* are defined by the equation

$$P_n(t) = \frac{1}{2^n n!} f_n^{(n)}(t),$$

where $f_n(t) = (t^2 - 1)^n$.

Following the next steps, show that $P_n(t)$ is an eigenvector associated with the eigenvalue $n(n+1)$ of the Sturm-Liouville operator T on the interval $[-1, 1]$ given by $T(f) = (pf')'$, where $p(t) = t^2 - 1$.

- Show that $(t^2 - 1)f_n'(t) = 2ntf_n(t)$;
- Differentiate both sides of the previous equation in order to get

$$(t^2 - 1)f_n^{(n+2)}(t) + 2t(n+1)f_n^{(n+1)}(t) + n(n+1)f_n^{(n)}(t) =$$

$$= 2ntf_n^{(n+1)}(t) + 2n(n+1)f_n^{(n)}(t);$$

- Recognize that the above equation can be written as

$$\left((t^2 - 1)P_n'(t)\right)' = n(n+1)P_n(t).$$