Midterm
Chapters 1, 2, and 3

Date: July 17, 1995
Time: >2 hours

Linear Algebra and Applications

(Mathematics 642:550:01 – Summer 1995)

Problem #1: (20pts) Consider the 5×5 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Write a decomposition $PA = LU$;

(b) Find a basis for the row space $\mathcal{R}$ of A;

(c) Describe which vectors $(c_1, c_2, c_3, c_4, c_5)$ are in the row space $\mathcal{R}$;

(d) Give an explicit description of the vector space $\mathcal{N}$ of all 5×1 column vectors X such that $AX = 0$;

(e) Find a basis for the nullspace $\mathcal{N}$;

(f) For what explicit 5×1 column vectors Y does the equation $AX = Y$ have solutions X?

Problem #2: (15pts) Using Gauss–Jordan elimination process, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}.$$

If $\det(A)$ is defined to be $\pm$ the product of the pivots of A ($\pm$ depends on the parity of the row exchanges you had to perform), what is $\det(A)$?

Problem #3: (10pts)

(a) Illustrate the notion of linear transformation. What else can you say if the vector spaces are finite dimensional?

(b) Let V be the vector space of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix. For any A in V define a map T by

$$T(A) = AB - BA.$$

- Verify that T is a linear transformation from V into V.
- What is the nullspace of T? Give some explicit examples of matrices that belong to the nullspace.
- What is $T^2(A) = T(T(A))$?

Problem #4: (10pts) Let V be the vector space of all functions from $\mathbf{R}$ to $\mathbf{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

(a) Prove that V_e and V_o are subspaces of V.

(b) Prove that $V_e + V_o = V$ (in Calc II, for example, you have seen that $e^x = (e^x + e^{-x})/2 + (e^x - e^{-x})/2 = \cosh x + \sinh x$).

(c) Prove that $V_e \cap V_o = \{0\}$.

Problem #5: (10pts)

(a) Define linearly (in)dependent sets; orthogonal sets; basis for a vector space; finite dimensional vector spaces. State some results that show the relationship among these concepts.

(b) Let V be the vector space of all $n \times n$ matrices over a field F. Why is it true that any matrix $A \in V = M_{n\times n}(F)$ satisfies a non-zero polynomial, i.e. there exists a polynomial $p(x) = c_o + c_1 x + \cdots + c_k x^k$ with non zero coefficients such that $p(A) = c_0 + c_1 A + \cdots + c_k A^k = 0$? Note that k depends on the matrix A.

Problem #6: (15pts) Let $\varphi(t)$ be a non-negative continuous real-valued function on the closed interval $[0,1]$ which is positive on $(0,1)$. For any pair of continuous real-valued functions f and g on $[0,1]$, define $\langle f, g\rangle = \int_0^1 f(t)g(t)\varphi(t)dt$.

(a) Show that $\langle f, g\rangle$ is an inner product on the vector space of all real-valued continuous functions on $[0,1]$.

(b) If $\varphi(t) = 2t$ for all $t \in [0,1]$, find the norm of $(t^2+1)^{-1}$ with respect to this inner product.

(c) If $\varphi(t) = 2t$ for all $t \in [0,1]$, find an orthogonal basis for the subspace P_2 of all polynomials of degree ≤ 2. Check that your polynomials are indeed orthogonal.

Problem #7: (10pts) Give $\mathbf{R}^3$ the standard inner product.

(a) Write the matrix P_W that projects every point in $\mathbf{R}^3$ onto the subspace $W = \{(x,y,z) \in \mathbf{R}^3 \mid x + y - z = 0\}$.

(b) Write the matrix $P_{W^\perp}$ that projects every point in $\mathbf{R}^3$ onto the subspace $W^\perp$.

Problem #8: (10pts)

(a) What is a least square solution $\overline{x}$ of a system $Ax = b$? Why is it the *best solution*?

(b) Solve $Ax = b$ by least squares and find $p = A\overline{x}$ if

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

Verify that the error $b - p$ is orthogonal to the columns of A.

The following three problems are intended only for EXTRA CREDIT. Look at them if and only if you have solved the previous eight problems.

Problem #P1: Let V be an n-dimensional vector space and let $T: V \longrightarrow V$ be a linear transformation. We have seen in class that if we fix a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ for V we can associate a matrix A to T. Also, if we consider another basis $\mathcal{B}' = \{v'_1, \dots, v'_n\}$ for V, the matrix A' that corresponds to T with respect to this new basis is related to A by the formula

$$A' = P^{-1}AP,$$

where $P = (p_{ij})$ is the invertible $n \times n$ matrix whose j-th column consists of the coordinates of $v'_j = p_{1j}v_1 + \cdots + p_{nj}v_n$.
Verify that the determinant and the trace are two functions that are invariant under change of coordinates, i.e.

(a) $\det(A) = \det(A')$;

(b) $\mathtt{trace}(A) = \mathtt{trace}(A')$.

Problem #P2: **Definition** *Let A be an $m \times n$ matrix with entries in $\mathbf{R}$; any $n \times m$ matrix A^+ with entries in $\mathbf{R}$ that satisfies the following four properties*

1. $AA^+A = A$;
2. $A^+AA^+ = A^+$;
3. $AA^+ = (AA^+)^{\mathrm{T}}$;
4. $A^+A = (A^+A)^{\mathrm{T}}$

is said to be a Moore–Penrose pseudoinverse of A.

We list some of the properties of the Moore–Penrose pseudoinverse of A. Prove as many properties as possible.

P1 A^+ is uniquely determined by conditions *1.*, *2.*, *3.*, and *4.*

P2 If A is a non-singular $n \times n$ matrix then $A^+ = A^{-1}$.

P3 If $\mathrm{rank}(A) = n$, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

P4 If $\mathrm{rank}(A) = m$, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$.

P5 If A is a projection matrix, then $A^+ = A$.

P6
- $\mathrm{rank}(A) = \mathrm{rank}(A^+)$;
- $(A^+)^+ = A$;
- $(A^+)^{\mathrm{T}} = (A^{\mathrm{T}})^+$;
- $(cA)^+ = c^{-1}A^+$ if $c \neq 0$;
- if $A = A^{\mathrm{T}}$ then $A^+ = (A^+)^{\mathrm{T}}$.

P7 For every matrix A the Moore–Penrose pseudoinverse A^+ always exists.

Problem #P3: **Definition** *Let T be a linear operator on an inner product space V. Then we say that T has an adjoint on V if there exists a linear operator T^* on V such that*

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle$$

for all v and w in V.

Remark It is a fact that every linear operator on a finite dimensional inner product space V has an adjoint on V. In the infinite dimensional case this is not always true. But in any case there is at most one such operator T^*; when it exists, we call it the *adjoint* of T.

(a) How can you view Problem #6 in terms of adjoints?

(b) Let $V = M_{n \times n}(\mathbf{R})$ with the inner product $\langle A, B \rangle = \mathtt{trace}(AB^{\mathrm{T}})$. Let M be a fixed $n \times n$ matrix over $\mathbf{R}$ and let L_M be the operator "left multiplication" by M, i.e. $L_M(A) = MA$. Show that the adjoint L_M^* of L_M is the operator "left multiplication" by M^{T}, i.e. $L_M^* = L_{M^{\mathrm{T}}}$.