MA561 – Modern Algebra
Homework set # 2
The due date is September 17 (Wednesday), 2008.

9. Let G be a finite group and let x be an element of G of order n. Prove that if n is odd, then
$$x = (x^2)^k$$
for some k.

10. Let $(G, \cdot)$ be a group and let x be an element of G of infinite order. Prove that
(i) $x^l = x^m$ if and only if $l = m$ for all $l, m \in \mathbb{Z}$, that is the elements $\{x^n \mid n \in \mathbb{Z}\}$ are all distinct;
(ii) the order of x^m is infinite for any $m \in \mathbb{Z} \setminus \{0\}$.

11. Let G be a finite group of even order. Define $t(G) = \{g \in G \mid g \neq g^{-1}\}$.
Show that $t(G)$ has an even number of elements. Conclude that G contains an element of order 2.

12. Let G be a group such that $(ab)^i = a^i b^i$ for three consecutive integers i and all $a, b \in G$. Prove that G is abelian.

13. Show that the group $((\mathbb{Z}/8\mathbb{Z})^\times, \cdot)$ is not cyclic.

14. Let $(G, \cdot)$ be a group with identity element 1 and let $a, b \in G$ be such that $ab = 1$. Show that $ba = 1$, thus b is the inverse of a.

15. Consider the map $f \colon \mathbb{Z}/5\mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z}$ given by $\overline{a} \mapsto \overline{3a - 2}$. Compute the inverse map g and write in the form $g(b) = \overline{\alpha b + \beta}$ for some $\alpha, \beta \in \{0, 1, 2, 3, 4\}$.

16. If x and g are elements of the group $(G, \cdot)$, prove that x and $g^{-1}xg$ have the same order. Deduce that ab and ba have the same order for all $a, b \in G$.