MA561 – Modern Algebra
Homework set # 3
The due date is September 26 (Friday), 2008.

17. Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is an homomorphism if and only if G is abelian.

18. Let $D_{2n} = \langle r, s \,|\, r^n = s^2 = 1, \quad rs = sr^{-1} \rangle$.

- ⋆ Show that if x is not a power of r then $rx = xr^{-1}$.
- ⋆ Show that any such x has order 2.
- ⋆ If $n = 2k$ is even and $n \geq 4$, show that r^k is an element of order 2 and that it commutes with all the elements of D_{2n}.

19. Prove that the order of $\mathrm{GL}_2(\mathbb{F}_2)$ is 6. Note that $\mathbb{F}_2 = \mathbb{Z}_2$.

20. Prove that a group G cannot have a subgroup H with $|H| = n-1$, where $n = |G| > 2$.

21. Let $H_1 \leq H_2 \leq \cdots$ be an ascending chain of subgroups of a group G. Prove that $\bigcup_{i=0}^{\infty} H_i$ is a subgroup of G.

22. Let G be an abelian group. Prove that $H = \{g \in G \,|\, |g| < \infty\}$ is a subgroup of G, called the *torsion subgroup* of G.

23. Let G be a finite group of order n. Use Lagrange's Theorem to show that the map

$$\gamma\colon\ G \longrightarrow G, \quad g \mapsto \gamma(g) = g^k$$

is surjective for any integer k relatively prime to n. That is, for such integer k any element $g \in G$ has a k^{th} root in G.

24. Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove *Fermat's Little Theorem*:

If p is a prime then $\quad a^p \equiv a \mod p \quad$ for all $a \in \mathbb{Z}$.