MA561 – Modern Algebra
Homework set # 4
The due date is October 8 (Wednesday), 2008.

25. ⋆ Let G be an abelian group and $H \leq G$ be a subgroup of G. Show that the quotient group G/H is abelian.

⋆ Give an example of a non-abelian group G and a normal subgroup H of G such that G/H is abelian.

26. Let G and H be groups. Let $\varphi\colon G \longrightarrow H$ be a surjective homomorphism and let N denote the kernel of φ. For $h \in H$ let $\varphi^{-1}(h) = \{g \in G \,|\, \varphi(g) = h\}$ be the fiber of h.

Show that for all $h \in H$ we have $\varphi^{-1}(h) = gN = Ng$, where g is any element in $\varphi^{-1}(h)$.

27. Find the cycle decomposition as well as the inverse of the permutation

$$\sigma = (7\ 5\ 1)(1\ 5\ 3\ 2)(4\ 1)(3\ 1\ 6)(7\ 1\ 3) \in S_7.$$

28. Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = \{e_G\}$.

29. Let G be a group and let $K \leq G$ and $H \trianglelefteq G$. Show that $H \cap K \trianglelefteq K$ and $HK \leq G$.

30. Let H and K be normal subgroups of G such that $G = HK$. Prove that $G/(H \cap K) \cong G/H \times G/K$. In particular, if $H \cap K = \{e_G\}$ one has that $G \cong G/H \times G/K$.

31. Let $G = \{1, a, b, c\}$ be a group of order 4. Show that either $G \cong \mathbb{Z}/4\mathbb{Z}$ or $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

32. Let G be a group. For any $g \in G$ define the map

$$f_g\colon G \longrightarrow G, x \mapsto f_g(x) = gxg^{-1},$$

called *inner automorphism* (or *conjugation*).

⋆ Verify that f_g is an automorphism of G.

⋆ Show that the map $\varphi\colon g \mapsto f_g$ is a homomorphism of G into $\mathrm{Aut}(G)$. Describe $\ker \varphi$.