MA561 – Modern Algebra
Homework set # 8
The due date is December 12 (Friday), 2008.

57. Let R be a finite commutative ring with identity. Prove that every prime ideal of R is a maximal ideal.

58. A commutative ring R is called a *principal ideal ring* if every ideal of R is principal. Show that if R is a principal ideal ring and I is any ideal of R then R/I is a principal ideal ring.

59. Find all solutions $x \in \mathbb{Z}$ to

$$x \equiv 4 \mod 12, \qquad x \equiv 6 \mod 35, \qquad x \equiv 2 \mod 11.$$

60. Consider the polynomial ring $\mathbb{Z}[x]$ and show that the ideal

$$(2, x) = \{2f + xg \mid f, g \in \mathbb{Z}[x]\}$$

is not principal.

61. Let R be a Boolean ring.

- $\star$ Prove that a nonzero Boolean ring has characteristic 2.
- $\star$ Prove that every prime ideal of R is a maximal ideal.
- $\star$ Prove that every finitely generated ideal of R is principal.
  (**Hint:** Start by showing that the ideal generated by two elements f and g equals the ideal generated by $f + g + fg$. Now repeat the process.)

62. Let R be a commutative ring with identity and with the property that for all $a \in R$ there exists a natural number $n \geq 2$ such that $a^n = a$. Show that every prime ideal is maximal.

63. Let R be a commutative ring with identity and let $P \subset R$ be a prime ideal in R. Let I, J be ideals in R such that $I \cap J \subseteq P$. Show that $I \subseteq P$ or $J \subseteq P$.

64. Let $\varphi\colon\ R \longrightarrow S$ be a homomorphism of commutative rings.

- $\star$ Prove that if Q is a prime ideal of S then either $\varphi^{-1}(Q) = R$ or $\varphi^{-1}(Q)$ is a prime ideal of R. In particular, if R is a subring of S and φ is the inclusion homomorphism deduce that if Q is a prime ideal of S then $Q \cap R = R$ or $Q \cap R$ is a prime ideal of R.
- $\star$ Prove that if M is a maximal ideal of S and φ is surjective then $\varphi^{-1}(M)$ is a maximal ideal of R.