THE KODAIRA DIMENSIONS OF $\overline{\mathcal{M}}_{22}$ AND $\overline{\mathcal{M}}_{23}$

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Abstract. We prove that the moduli spaces of curves of genus 22 and 23 are of general type. To do this, we calculate certain virtual divisor classes of small slope associated to linear series of rank 6 with quadric relations. We then develop new tropical methods for studying linear series and independence of quadrics and show that these virtual classes are represented by effective divisors.

Contents

1. Introduction

Many of the familiar moduli spaces in algebraic geometry, such as those parametrizing curves, abelian varieties, or $K3$ surfaces, have infinitely many irreducible components, of which all but finitely many are of general type. The remaining few components are typically uniruled. Understanding which components are uniruled and which are of general type is often difficult. Indeed, aside from the moduli of spin curves [FV14], all of the standard moduli spaces include notorious open cases, that is, components whose Kodaira dimensions are not known. The aim of the present paper is to resolve two long-standing cases for the moduli space $\overline{\mathcal{M}}_g$ of curves of genus g.

Theorem 1.1. *The moduli spaces $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type.*

This extends earlier results of Harris, Mumford and Eisenbud, who showed that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$, in the landmark papers [HM82, Har84, EH87], and improves on the thesis result of the first author, who showed that $\overline{\mathcal{M}}_{23}$ has Kodaira dimension at least 2 [Far00]. These general type statements contrast with the classical result of Severi [Sev15] that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$ (see [AC81] for a modern treatment) and with the more recent results of many authors [Ser81, CR84, CR86, Ver05, BV05, Sch15], which taken together show that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 14$ and that $\overline{\mathcal{M}}_{15}$ is rationally connected. Chang and Ran also argued that $\overline{\mathcal{M}}_{16}$ is uniruled [CR91], but Tseng recently found a fatal computational error in this argument [Tse19], and this case is again open. The Kodaira dimension of $\overline{\mathcal{M}}_g$ in unknown for $16 \leq g \leq 21$.

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1.1. **Divisors of small slope.** As in the earlier proofs for $g \geq 24$, we show that $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type by producing effective divisors of slope less than $\frac{13}{2}$, which is the slope of the canonical divisor $K_{\overline{\mathcal{M}}_g}$. The Slope Conjecture of Harris and Morrison [HM90] predicted that all effective divisors on $\overline{\mathcal{M}}_g$ have slope at least $6 + \frac{12}{g+1}$. This led people to believe that $\overline{\mathcal{M}}_g$ would be uniruled for $g < 23$. The earliest known counterexample to the Slope Conjecture is the closure in $\overline{\mathcal{M}}_{10}$ of the locus of smooth curves lying on a $K3$ surface, which is equal to the divisorial component of the locus of curves $[X] \in \mathcal{M}_{10}$ with a degree 12 map to $\mathbf{P}^4$ whose image is contained in a quadric [FP05]. This is the first in an infinite sequence of counterexamples in genus $2s^2 + s$, for $s \geq 2$; the second is the closure in $\overline{\mathcal{M}}_{21}$ of the divisorial component of the locus of curves with a degree 24 map to $\mathbf{P}^6$ whose image is contained in a quadric [Far09, Kho07].

Our divisors of small slope on $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are natural generalizations of this second example on $\overline{\mathcal{M}}_{21}$. Roughly speaking, the divisors $\widetilde{\mathfrak{D}}_{22}$ and $\widetilde{\mathfrak{D}}_{23}$ are the closures of the loci of smooth curves with a map to $\mathbf{P}^6$ of degree 25 and 26, respectively, whose image is contained in a quadric. We note, however, that this example on $\overline{\mathcal{M}}_{21}$ is the push forward of a codimension 1 locus in a space of linear series that is generically finite over the moduli of curves. A major new difficulty in the present construction is that we push forward a higher codimension locus in a space of linear series that maps onto $\overline{\mathcal{M}}_g$ with positive dimensional fibers. This makes both carrying out the intersection theory calculations and checking the needed transversality assumptions incomparably more challenging. We now sketch the construction; for the precise details, see §3.

For a general curve X of genus $g = 22$ or 23, the variety $W^6_{g+3}(X)$ is irreducible of dimension equal to $g - 21$. Moreover, each line bundle $L \in W^6_{g+3}(X)$ is very ample, with $h^0(X, L) = 7$ and $h^1(X, L) = 3$. Consider the multiplication map

$$\phi_L \colon \mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2}).$$

Note that $\dim \mathrm{Sym}^2 H^0(X, L) = 28$ and, by Riemann-Roch, $h^0(X, L^{\otimes 2}) = g + 7$. Therefore, the locus where ϕ_L is non-injective has expected codimension $h^0(X, L^{\otimes 2}) - 28 + 1 = g - 20$ in the space of such pairs $[X, L]$. Since this expected codimension is one more than the dimension of $W^6_{g+3}(X)$, one expects its image in $\overline{\mathcal{M}}_g$, which is the locus of curves with a map of degree $g + 3$ to $\mathbf{P}^6$ with image contained in a quadric, to have codimension 1. To use the closure of this locus to prove that $\overline{\mathcal{M}}_g$ is of general type for $g = 22$ and 23, there are three significant challenges: (i) computing the expected slope, by pushing forward the virtual class of the degeneracy locus for a natural map of vector bundles whose fiber over $[X, L]$ is ϕ_L, (ii) showing the closure of this locus is not all of $\overline{\mathcal{M}}_g$, and (iii) showing that the push forward of the virtual class is effective. The next theorem concerns the computation of the expected slope.

We work over an open substack $\widetilde{\mathfrak{M}}_g$ of the moduli stack of stable curves $\overline{\mathfrak{M}}_g$, whose rational divisor class group is freely generated by the Hodge class λ and the boundary classes δ_0 and δ_1. We then consider a stack of limit linear series $\sigma \colon \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathfrak{M}}_g$, where $r = 6$ and $d = g + 3$, and a map of vector bundles over $\widetilde{\mathfrak{G}}^r_d$ that restricts to ϕ_L over $[X, L]$. The locus $\mathfrak{U}$ of pairs $[X, L]$ where ϕ_L is not injective inherits a closed determinantal substack structure, as a degeneracy locus for this map of vector bundles, and hence it carries a virtual class of expected codimension $g - 20$. Let $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$ be the push forward of this virtual class, which is a divisor class on $\widetilde{\mathfrak{M}}_g$.

Theorem 1.2. *The virtual divisor classes $[\widetilde{\mathfrak{D}}_{22}]^{\mathrm{virt}}$ and $[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}}$ associated to the loci of curves of genus 22 and 23 with maps to $\mathbf{P}^6$ of degree 25 and 26 with image contained in a quadric are*

$$[\widetilde{\mathfrak{D}}_{22}]^{\mathrm{virt}} = \frac{2}{3}\binom{19}{8}\Big(17121\lambda - 2636\ \delta_0 - 14511\ \delta_1\Big) \in CH^1(\widetilde{\mathfrak{M}}_{22})$$

and respectively

$$[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}} = \frac{4}{9}\binom{19}{8}\Big(470749\lambda - 72725\ \delta_0 - 401951\ \delta_1\Big) \in CH^1(\widetilde{\mathfrak{M}}_{23}).$$

For the precise definitions of $\widetilde{\mathfrak{M}}_g$, the stack $\widetilde{\mathfrak{G}}^r_d$ of limit linear series and the virtual classes $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$, we refer the reader to §3. Provided that these virtual classes are represented by effective divisors, pushing forward to the coarse space and then taking closures in $\overline{\mathcal{M}}_g$ produces divisor classes of slope $\frac{17121}{2636} = 6.495\ldots$ and $\frac{470749}{72725} = 6.473\ldots$ in $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$, respectively. Most importantly for the proof of Theorem 1.1, both of these slopes are strictly less than $\frac{13}{2}$.

This construction is inspired by results in [FP05], where it is shown that any divisor on $\overline{\mathcal{M}}_g$ with slope less that $6 + \frac{12}{g+1}$ must contain the locus $\mathcal{K}_g \subseteq \mathcal{M}_g$ of curves lying on a $K3$ surface. Finding geometric divisors on $\overline{\mathcal{M}}_g$ which contain this locus has proven to be quite difficult, as curves on $K3$ surfaces behave generically with respect to many natural geometric properties, such as Brill-Noether and Gieseker-Petri conditions.

1.2. **Strong Maximal Rank Conjecture.** The Maximal Rank Conjecture, now a theorem of Larson [Lar17], has classical origins in the work of M. Noether and Severi [Sev15]. It was brought to modern attention by Harris [Har82]. It says that if X is a general curve of genus g and $L \in W^r_d(X)$ is a general linear series, then the multiplication of global sections

$$\phi^k_L \colon \mathrm{Sym}^k H^0(X, L) \to H^0(X, L^{\otimes k})$$

is of maximal rank for all k. This determines the Hilbert function of the general embedding of the general curve for each degree and genus. The Maximal Rank Conjecture has been the focus of much activity over the decades, with many important cases, especially for small values of k, proved using embedded degenerations in projective space [BE89, BF10], tropical geometry [JP16, JP17], or limit linear series [LOTiBZ17]. These special cases have applications, including to the surjectivity of Wahl maps [Voi92] and the construction of counterexamples to the Slope Conjecture [FP05].

The *Strong Maximal Rank Conjecture* is a proposed refinement that takes into account *every* linear series $L \in W^r_d(X)$ on a general curve, rather than just the *general* one [AF11, Conjecture 5.4]. The case $k = 2$ is of particular interest, because the failure of the map $\phi_L := \phi^2_L$ to be of maximal rank is equivalent to the existence of a rank 2 vector bundle with a prescribed number of sections, and it is known due to work of Lazarsfeld and Mukai that this is a condition that distinguishes curves lying on $K3$ surfaces. It predicts that for a general curve X of genus g, and for positive integers r, d such that $0 \le \rho(g, r, d) \le r - 2$, the determinantal variety

$$\Sigma^r_d(X) := \Big\{ L \in W^r_d(X) : \phi_L \text{ is not of maximal rank} \Big\}$$

has the expected dimension. In particular, the Strong Maximal Rank Conjecture predicts that ϕ_L is injective for *every* line bundle $L \in W^r_d(X)$ when the following inequality is satisfied:

$$\text{expdim } \Sigma^r_d(X) := g - (r+1)(g - d + r) - (2d + 1 - g) + \frac{r(r+3)}{2} < 0. \tag{1}$$

When expdim $\Sigma^r_d(X) = -1$, the locus of curves for which $\Sigma^r_d(X)$ is not empty has expected codimension 1 in $\mathcal{M}_g$, and contains the locus of curves on $K3$ surfaces. So its divisorial part is a natural candidate for an effective divisor of small slope. In the two cases $g = 22$, $d = 25$, $r = 6$ and $g = 23$, $d = 26$ and $r = 6$, the Strong Maximal Rank Conjecture amounts to the statement that the degeneracy locus $\mathfrak{U}$ discussed above does not dominate $\overline{\mathcal{M}}_g$, so its divisorial part is well-defined. We prove the conjecture in these two cases.

Theorem 1.3. *Set $g = 22$ or 23. For a general curve X of genus g, the multiplication map*

$$\phi_L \colon \mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$$

is injective for all *line bundles $L \in W^6_{g+3}(X)$.*

Theorem 1.3 shows that the determinantal locus $\mathfrak{U}$ does not map dominantly onto $\overline{\mathcal{M}}_g$. It follows that $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$ is a divisor, rather than just a divisor class. In other words, the virtual class is a linear combination of the codimension 1 components of the image of $\mathfrak{U}$ in $\widetilde{\mathcal{M}}_g$. The first proof of

Theorem 1.3 appeared in the preprint [JP18]; that work was never submitted for publication and is incorporated into the present paper. An alternative approach using limit linear series was put forward in [LOTiBZ18].

The main difficulty in the proof of Theorem 1.3, in comparison with the corresponding cases of the Maximal Rank Conjecture, is that one must control *all* linear series on the general curve X, rather than just a sufficiently general one. For this purpose, the embedded degeneration methods initiated by Hartshorne, Hirschowitz and much refined by Larson are not suitable. Instead, we prove Theorem 1.3 by taking X to be a curve over a nonarchimedean field whose skeleton is a chain of loops with specified edge lengths and applying tropical methods to study the linear series of degree $g+3$ and rank 6. Along the way, we develop new techniques for understanding the tropicalization of a linear series, based on the valuated matroids given by relations among collections of sections (see, e.g., Example 6.8), and an effective criterion for verifying tropical independence (Theorem 1.6). Each of these represents a significant advance beyond the approach to maximal rank statements via tropical methods developed in [JP16, JP17].

1.3. **Effectivity of the virtual divisor.** Together, Theorems 1.2 and 1.3 do not suffice to show that $[\widetilde{\mathfrak{D}}_g]^{\text{virt}}$ is effective on $\widetilde{\mathcal{M}}_g$. Theorem 1.3 does establish that for $g = 22$ or 23, the image of the degeneracy locus $\mathfrak{U}$ has positive codimension. Since the push forward of its virtual class is well-defined as a divisor class supported on its image, it follows that

$$[\widetilde{\mathfrak{D}}_g]^{\text{virt}} = a_1\mathcal{Z}_1 + \cdots + a_s\mathcal{Z}_s$$

is a linear combination of the codimension one components in the image of $\mathfrak{U}$. A priori, the degeneracy locus $\mathfrak{U}$ could still have components of higher than expected dimension that map with positive dimensional fibers onto some of these codimension 1 components, in which case, some coefficient a_i may be negative. The following theorem rules out this possibility.

Theorem 1.4. *Let $\mathcal{Z} \subseteq \overline{\mathcal{M}}_g$ be the closure of a codimension one component of $\sigma(\mathfrak{U})$. Then the generic fiber of $\mathfrak{U}$ over $\mathcal{Z}$ is finite.*

The proof of Theorem 1.4 has two main parts. One part, carried out in §10, uses tropical methods, very similar to those used in the proof of Theorem 1.3, to show that if the generic fiber of $\mathfrak{U}$ over $\mathcal{Z}$ is infinite, then $\mathcal{Z}$ does not contain certain codimension 2 strata in $\overline{\mathcal{M}}_g$, and to control its pull back under natural maps between moduli spaces. More precisely, we show that the class $[\mathcal{Z}]$ pulls back to *zero* under the map $\jmath_2 \colon \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_g$ obtained by attaching a general pointed curve of genus $g-2$ to each pointed curve of genus 2. Similarly, we show that $[\mathcal{Z}]$ pulls back to a nonnegative combination of the Weierstrass divisor and the hyperelliptic divisor in $\overline{\mathcal{M}}_{3,1}$ under the analogous attaching map $\jmath_3 \colon \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}_g$. The other part, carried out in §2.3, is a series of computations in $CH^1(\overline{\mathcal{M}}_g)$ showing that there is no nonzero effective divisor with these properties.

Theorem 1.1 follows in a straightforward manner from Theorems 1.2, 1.3, and 1.4. Indeed, Theorems 1.3 and 1.4 together imply that $[\widetilde{\mathfrak{D}}_g]$ is a well-defined effective divisor on $\widetilde{\mathcal{M}}_g$. Taking closure in $\overline{\mathcal{M}}_g$, for $g = 22$ or 23 gives an effective divisor whose slope is the ratio $\frac{a}{b_0}$ computed in Theorem 1.2. It follows that $\overline{\mathcal{M}}_g$ is of general type, since this slope is less than $\frac{13}{2}$.

1.4. **Tropical independence.** Our proofs of Theorems 1.3 and 1.4 are based on tropical independence, as in [JP16, JP17]. Roughly speaking, tropical independence is a method for proving that a set of sections $\{s_1, \ldots, s_n\}$ of a line bundle is linearly independent by extending the line bundle and the sections over a semistable degeneration such that the specialization map is diagonal, i.e., there are irreducible components $X_1, \ldots, X_n$ in the special fiber such that s_i is nonzero on X_j if and only if $i = j$. We now briefly summarize the foundations of the method, with the hope that this will be helpful for those accustomed to other degeneration techniques in the study of algebraic curves and their linear series.

1.4.1. *Tropicalization of rational functions.* Let X be a curve over a complete and algebraically closed valued field K, with valuation ring R. Suppose $\mathcal{X}$ is a semistable model of X, that is, a flat and proper scheme over $\operatorname{Spec} R$ with generic fiber $\mathcal{X}_K \cong X$ and a reduced special fiber with only nodal singularities. Near each node in the special fiber, $\mathcal{X}$ is étale locally isomorphic to $xy = f$ for some f in the maximal ideal of R. The valuation $\operatorname{val}(f)$ is independent of the choice of coordinates, and is called the *thickness* of the node.

Let Γ be the metric dual graph of this degeneration. The underlying graph has one vertex for each irreducible component of the special fiber and one edge for each node. Loops and multiple edges appear when irreducible components have self-intersections and when two components meet at multiple nodes. The length of an edge is the thickness of the corresponding node.

Each point v in Γ is naturally identified with a valuation val_v on the function field $K(X)$. Roughly speaking, the valuation at a vertex corresponds to the order of vanishing along the corresponding component of the special fiber. More precisely, $\operatorname{val}_v(f)$ is equal to $-\operatorname{val}(a)$ for any $a \in K^*$ such that af is regular and nonvanishing at the generic point of the corresponding component X_v. The points in the interior of an edge correspond to monomial valuations in the local coordinates x and y at a node $xy = f$ that agree with the given valuation on the scalar subfield $K \subseteq K(X)$.

Remark 1.5. In the special case where X is defined over a discretely valued subfield $K' \subseteq K$ and $\mathcal{X}$ is defined over the valuation ring $R' \subseteq K'$, the thickness of each node is an integer, and $\mathcal{X}$ has an A_{n-1} singularity at a node of thickness n. Recalling that the length of an edge equals the thickness of the corresponding node, the valuations given by the integer points on the corresponding edge of length n are the vanishing orders along the $n-1$ exceptional components of the chain of rational curves in the minimal resolution of this singularity.

Each valuation is naturally identified with a point in the Berkovich analytification X^{an}, and the resulting map $\Gamma \to X^{\mathrm{an}}$ is a homeomorphism onto its image (and, in an appropriate sense, an isometry). When no confusion seems possible, we identify Γ with its image in X^{an}. There is a natural retraction $\operatorname{Trop} : X^{\mathrm{an}} \to \Gamma$, which is called *tropicalization*, as is the induced map $\operatorname{Div}(X) \to \operatorname{Div}(\Gamma)$ taking a formal sum of K-points to the formal sum of their images in Γ.

The tropicalization of a nonzero rational function $f \in K(X)^*$ is defined as

$$\operatorname{trop}(f) \colon \Gamma \to \mathbb{R}; \quad v \mapsto \operatorname{val}_v(f).$$

This function is continuous, and piecewise linear on each edge, with integer slopes. Moreover, $\operatorname{trop}(f)$ is determined up to an additive constant by $\operatorname{Trop}(\operatorname{div}(f))$.

1.4.2. *Tropicalization of linear relations.* Suppose $\{f_1, ..., f_n\} \subseteq K(X)^*$ is a collection of non-zero rational functions satisfying a linear relation $a_1 f_1 + \cdots + a_n f_n = 0$, with $a_i \in K^*$. Note that $\operatorname{val}_v(a_i f_i) = \operatorname{trop}(f_i)(v) + \operatorname{val}(a_i)$. Therefore, at each point $v \in \Gamma$, the minimum in

$$\theta(v) = \min_i \{\operatorname{trop}(f_i)(v) + \operatorname{val}(a_i)\}$$

must be achieved at least twice. This is a strong restriction on the functions $\operatorname{trop}(f_i)$. For instance, after subdividing Γ so that each of these functions has constant slope on every edge, then on each edge there must be two functions with equal slope.

We say that a collection of piecewise linear functions with integer slopes $\{\psi_1, ..., \psi_n\} \in PL(\Gamma)$ is *tropically dependent* if there are real numbers $b_1, \ldots, b_n$ such that, for every point $v \in \Gamma$, the minimum in $\min_i\{\psi_i(v) + b_i\}$ is achieved at least twice. If there are no such real numbers, then we say that $\{\psi_1, \ldots, \psi_n\}$ is *tropically indepedent*. Tropical independence of $\{\operatorname{trop}(f_1), \ldots, \operatorname{trop}(f_n)\}$ is a sufficient condition for the linear independence of $\{f_1, \ldots, f_r\} \subseteq K(X)^*$.

Now suppose that $L = \mathcal{O}_X(D_X)$ is a line bundle in $W^r_d(X)$, as in §1.1 above. We identify $H^0(X, L)$ and $H^0(X, L^{\otimes 2})$ with K-linear subspaces of $K(X)$, in the usual way. We can then show that the map $\phi_L \colon \operatorname{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$ has rank at least n by finding rational functions $f_1, \ldots, f_n$ in the image of ϕ_L such that $\{\operatorname{trop}(f_1), \ldots, \operatorname{trop}(f_n)\}$ is tropically independent. In practice, we do not work directly with rational functions in the image of ϕ_L. Instead, we identify piecewise linear functions $\varphi_1, \ldots, \varphi_s$ in the image of $H^0(X, L)$ under tropicalization. Then all pairwise sums $\varphi_{ij} = \varphi_i + \varphi_j$ are tropicalizations of functions in the image of ϕ_L. To prove that ϕ_L is injective, we look for a set of such pairwise sums, of size equal to $\dim \operatorname{Sym}^2 H^0(X, L)$, that is tropically independent.

1.4.3. *A characterization of tropical independence.* One of the foundational advances in this paper is a new necessary and sufficient condition for tropical independence. Given a finite set of PL functions $\{\varphi_1, \ldots, \varphi_n\}$ on Γ, and real numbers $b_1, \ldots, b_n$, we consider the corresponding *tropical linear combination*

$$\theta = \min_i \{\varphi_i + b_i\}.$$

We say that φ_i *achieves the minimum* at v if $\theta(v) = \varphi_i(v) + b_i$ and that it *achieves the minimum uniquely* if, furthermore, $\theta(v) \neq \varphi_j(v) + b_j$ for $j \neq i$.

Theorem 1.6. *A finite set of PL functions $\{\varphi_1, \ldots, \varphi_n\}$ on Γ is tropically independent if and only if there are real numbers $b_1, \ldots, b_n$ such that each φ_i achieves the minimum uniquely at some $v \in \Gamma$.*

This is proved in §2.5, using the Knaster-Kuratowski-Mazurkiewicz lemma, a set-covering variant of the Brouwer fixed-point theorem.

1.4.4. *From tropical independence to diagonal specialization.* We now return to the setup where X is a curve over a valued field, $\mathcal{X}$ is a semistable model with metric dual graph Γ, $L = \mathcal{O}_X(D_X)$ is a line bundle, and $f_1, \ldots, f_r \in K(X)^*$ are sections of L. Let $\varphi_i = \operatorname{trop}(f_i)$. Let $\theta = \min_i \{\varphi_i + b_i\}$ be a tropical linear combination in which each φ_i achieves the minimum uniquely at some point $v_i \in \Gamma$.

We can then choose a toroidal modification $\mathcal{X}' \to \mathcal{X}$ so that each v_i corresponds to an irreducible component X_i of the special fiber of $\mathcal{X}'$, and each of the functions $\varphi_1, \ldots, \varphi_r$ and θ is linear on each edge of the metric dual graph Γ', which is a subdivision of Γ. Furthermore, we can extend L to a line bundle $\mathcal{L}$ over $\mathcal{X}'$ so that $f \in H^0(X, L)$ is a regular section of $\mathcal{L}$ if and only if $\operatorname{trop}(f) \geq \theta$, and nonvanishing on the component X_v corresponding to a vertex $v \in \Gamma'$ if and only if $\operatorname{trop}(f)(v) = \theta(v)$; see Proposition 6.6. In particular, if $a_i \in K^*$ are scalars such that $\operatorname{val}(a_i) = b_i$, then $a_i f_i$ is a regular section of $\mathcal{L}$ and is nonvanishing on the irreducible component corresponding to v_j if and only if $i = j$. This diagonal specialization property ensures that $\{a_1 f_1, \ldots, a_r f_r\}$ is independent in the special fiber, and hence also independent in the general fiber.

1.5. **Chains of loops.** In our proofs of Theorems 1.3 and 1.4, we apply the method of tropical independence (and Theorem 1.6 in particular) to linear series on curves X whose skeletons are specific, carefully chosen graphs Γ. As in [JP16, JP17], the graphs Γ are chains of loops with specified edge lengths. The divisor classes on such graphs Γ that can arise in tropicalizations of linear series of degree d and dimension r have been studied in [CDPR12, Pfl17, JR17, CPJ19]. For those unfamiliar with such curves, we explain the geometry of their stable reductions.

Let X be a curve of genus g over K whose skeleton is a chain of g loops Γ. Let X_0 be its stable reduction. Then $[X_0] \in \overline{\mathcal{M}}_g$ is a 0-stratum. One can label its $2g-2$ rational components as

$$X_1, Y_2, X_2, Y_3, \ldots, X_{g-1}, Y_g,$$

such that

(1) The components X_1 and Y_g each have one node, and the rest are smooth;
(2) For $1 \leq i \leq g-1$, X_i meets Y_{i+1} at a single node;
(3) For $2 \leq i \leq g-1$, X_i meets Y_i at two nodes.

This curve X_0 is in the closure of the locus of hyperelliptic curves in $\mathcal{M}_g$. Our arguments therefore cannot use the geometry of X_0 in any meaningful way. Instead, we use the edge lengths of Γ. Thinking of X as the general fiber in a family over a germ of a curve, with central fiber X_0, the edge lengths specify the contact orders of this curve with the $3g-3$ branches of the boundary divisor that meet at $[X_0]$. Our proof of Theorem 1.3 shows that the general member of such a family with certain specified contact orders satisfies the conclusion of the Strong Maximal Rank Conjecture.

For those familiar with this method, we briefly describe the novel aspects of the constructions presented here. Recall that the space of all divisor classes of degree d on Γ is a real torus of dimension g, and the subspace $W^r_d(\Gamma)$ parametrizing those that can come from linear series of degree d and rank r on an algebraic curve form a finite union of translates of subtori, called *combinatorial types*. These combinatorial types are naturally indexed by certain tableaux. When the edge lengths of Γ are sufficiently general, an open dense subset of $W^r_d(\Gamma)$ consists of divisor classes that are *vertex avoiding*, in the sense of [CJP15].

Suppose X is a curve whose tropicalization is Γ, D_X is a divisor of degree d in a linear series of dimension r, and assume the class of $D = \mathrm{Trop}(D_X)$ is vertex avoiding. Then we have a canonical collection of PL functions $\{\varphi_0, \dots, \varphi_6\}$ on Γ that is the tropicalization of a basis for $H^0(X, \mathcal{O}(D_X))$. In [JP16], we fix one particular D, assume that there is a tropical linear combination $\theta = \min_{ij}\{\varphi_i + \varphi_j + b_{ij}\}$ such that the minimum is achieved at least twice at every point $v \in \Gamma$, compute the degree of the associated effective divisor $D + \mathrm{div}(\theta)$, and derive a contradiction. There are several difficulties in extending this approach to *all* divisors of degree d and rank r. One is sheer combinatorial complexity. The arguments in [JP16] are specific to the combinatorial type of D. When Γ has genus 23, the number of combinatorial types in $W^6_{26}(\Gamma)$ is

$$\frac{23!}{9! \cdot 8! \cdot 7!} = 350,574,510.$$

This difficulty is overcome primarily through the new constructive method for proving tropical independence, given by Theorem 1.6.

In the non vertex avoiding cases, we face the additional problem of understanding which functions in $R(D)$ are tropicalizations of functions in $H^0(X, \mathcal{O}(D_X))$, and finding a suitable substitute for the distinguished functions φ_i. For an arbitrary divisor D, this seems to be an intractable problem. However, when ρ is small, we find that in most cases it is enough to understand the tropicalizations of certain pencils in $H^0(X, \mathcal{O}(D_X))$. These, in turn, behave similarly to tropicalizations of pencils on $\mathbf{P}^1$, which we analyze in §6.3. The possibilities for the tropicalizations of $H^0(X, \mathcal{O}(D_X))$ are then divided into cases, according to the combinatorial properties of these pencils. We then construct a tropical independence case-by-case, in §§8-9, using a generalization of the algorithm that works for vertex avoiding divisors. Only one subcase, treated in §9.4.4, does not reduce to an analysis of pencils; the arguments in this subcase are nevertheless of a similar flavor, with a few more combinatorial possibilities to consider.

1.6. **Further constructions of virtual divisors of small slope.** The construction of the virtual divisor class $[\widetilde{\mathfrak{D}}_{22}]$ and the computation of its slope can be extended to an infinite family of cases, as follows. Fix an integer $s \geq 2$, and set

$$g := 2s^2 + s + 1, \ d := 2s^2 + 2s + 1 \ \text{ and } \ r := 2s.$$

Then a general curve $[X] \in \mathcal{M}_g$ carries a 1-dimensional family of line bundles $L \in W^r_d(X)$. For each such L we consider the multiplication map

$$\phi_L \colon \mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2}).$$

Observe that $h^0(X, L^{\otimes 2}) - \dim \mathrm{Sym}^2 H^0(X, L) = 1$. Therefore, the locus where the map ϕ_L is non-injective has expected codimension *two* in the parameter space of pairs $[X, L]$, where $L \in W^r_d(X)$.

Just as in the special case $s = 3$ and $g = 22$ discussed above, we work over a partial compactification $\widetilde{\mathfrak{M}}_g$ of an open substack of $\overline{\mathfrak{M}}_g$ whose divisor class group is generated by λ, δ_0, and δ_1. We consider the stack $\sigma \colon \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathfrak{M}}_g$ of limit linear series of type $\mathfrak{g}^r_d$. On the resulting stack $\widetilde{\mathfrak{G}}^r_d$ we then construct a map of vector bundles that restricts to ϕ_L on the fiber over $[X, L]$, and we compute the push forward of the virtual class of the degeneracy locus $\mathfrak{U}$, where this map is not injective.

Theorem 1.7. *Fix $s \geq 2$ and set $g := 2s^2 + s + 1$. Let $\mathfrak{U} \subseteq \widetilde{\mathfrak{G}}^r_d$ be the degeneracy locus described above, and let $[\widetilde{\mathfrak{D}}_g] = \sigma_*[\mathfrak{U}]^{\mathrm{virt}}$. Write $[\widetilde{\mathfrak{D}}_g] = a\lambda - b_0\delta_0 - b_1\delta_1$. Then*

$$\frac{a}{b_0} = \frac{3(48s^8 - 56s^7 + 92s^6 - 90s^5 + 86s^4 + 324s^3 + 317s^2 + 182s + 48)}{24s^8 - 28s^7 + 22s^6 - 5s^5 + 43s^4 + 112s^3 + 100s^2 + 50s + 12}$$
$$= 6 + \frac{12}{g+1} - \frac{3(120s^6 - 140s^5 - 162s^4 + 67s^3 + 153s^2 + 94s + 24)}{(2s^2+s+1)(24s^8 - 28s^7 + 22s^6 - 5s^5 + 43s^4 + 112s^3 + 100s^2 + 50s + 12)}.$$

In particular, $\frac{a}{b_0} < 6 + \frac{12}{g+1}$ for all $s \geq 3$.

Setting $s = 3$, we obtain the virtual divisor $\widetilde{\mathfrak{D}}_{22}$ in $\widetilde{\mathcal{M}}_{22}$ that appears in Theorem 1.2. When $s = 2$, $\sigma_*[\mathfrak{U}]^{\mathrm{virt}}$ is an effective divisor whose closure in $\overline{\mathcal{M}}_{11}$ has slope 7. This interesting divisor has also appeared in [FO12, BF18], and can be seen as the closure of the locus of curves $[X] \in \mathcal{M}_{11}$ possessing a semistable rank 2 vector bundle E whose Clifford index $\mathrm{Cliff}(E)$ is strictly less than the Clifford index $\mathrm{Cliff}(X)$, which is, as usual, computed with respect to special line bundles on X.

For $g = 22$, results from [FP05] show that the closure in $\overline{\mathcal{M}}_g$ of the image of any effective representative of $[\mathfrak{U}]^{\mathrm{virt}}$ has slope equal to $\frac{a}{b_0}$. It is possible to extend this statement for $s > 3$, following closely the methods of [Far06]. However, in the interest of not increasing further the length of this paper we choose not to carry this out here.

We have written this paper for an audience that includes experts on moduli spaces as well as experts on tropical and nonarchimedean geometry. While the class computations (§4-5) and the main tropical arguments (§8-10) are necessarily technical, the presentation also includes detailed examples (such as Examples 6.8 and 7.2) and complete arguments in special cases, such as the proof of injectivity of ϕ_L in the vertex avoiding case, in §7. These are not logically necessary, but should clarify and motivate the essential steps in the proofs of the main theorems.

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2. Preliminaries

In this section we lay the groundwork for the main sections of the paper, establishing notation and recalling basic facts that will be used throughout, and proving two foundational results that may be more broadly useful. In particular, we recall the notion of slopes of divisors on moduli spaces of stable curves, review the intersection numbers for curves and divisors on these moduli spaces, and prove a vanishing criterion for effective divisors (Proposition 2.2). We then discuss skeletons of curves over nonarchimedean fields and tropicalization of rational functions, before proving an effective criterion for tropical independence (Theorem 1.6).

2.1. **Slopes of divisors.** We denote by $\overline{\mathfrak{M}}_g$ the moduli stack of stable curves of genus $g \geq 2$ and by $\overline{\mathcal{M}}_g$ the associated coarse moduli space. All of the cycles and Chow groups that we consider are with rational coefficients. The push forward $CH^*(\overline{\mathfrak{M}}_g) \to CH^*(\overline{\mathcal{M}}_g)$ is an isomorphism, and we identify each cycle and cycle class on $\overline{\mathfrak{M}}_g$ with its push forward to $\overline{\mathcal{M}}_g$. In particular, if $\mathfrak{Y} \subseteq \overline{\mathfrak{M}}_g$ is an irreducible closed substack with coarse moduli space $\mathcal{Y}$ and generic stabilizer G, then $[\mathfrak{Y}] = \frac{1}{|G|}[\mathcal{Y}]$.

All intersection theory calculations in this paper are carried out on the stack $\overline{\mathfrak{M}}_g$, whereas the results about Kodaira dimension concern the coarse space $\overline{\mathcal{M}}_g$. We follow the standard convention that CH^i denotes the Chow group of cycles of codimension i, modulo rational equivalence.

Recall that for $g \geq 3$ the group $CH^1(\overline{\mathcal{M}}_g) \cong CH^1(\overline{\mathfrak{M}}_g)$ is freely generated by the Hodge class λ and the classes of boundary divisors $\delta_i = [\Delta_i]$, for $i = 0, \ldots, \lfloor \frac{g}{2} \rfloor$. For $g = 2$ one has the supplementary relation $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$. The canonical class of $\overline{\mathcal{M}}_g$, computed in [HM82], is

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor \frac{g}{2} \rfloor}.$$

The singularities of $\overline{\mathcal{M}}_g$ are mild enough that all sections of $nK_{\overline{\mathcal{M}}_g}$ extend to pluricanonical forms on a resolution of singularities [HM82]. Therefore, $\overline{\mathcal{M}}_g$ is of general type if and only if there is an effective divisor class $[D] \sim a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i\delta_i$ on $\overline{\mathcal{M}}_g$ that satisfies

$$\frac{a}{b_i} < \frac{13}{2} \text{ for } i \neq 1 \text{ and } \frac{a}{b_1} < \frac{13}{3}. \tag{2}$$

The *slope* of an effective divisor D on $\overline{\mathcal{M}}_g$ with class as above is $s(D) := \max_i \left\{ \frac{a}{b_i} \right\}$ [HM90]. For the purpose of studying the Kodaira dimension of $\overline{\mathcal{M}}_g$, one is most concerned with $\frac{a}{b_0}$. Indeed, if $\frac{a}{b_0} < \frac{13}{2}$, then $\frac{a}{b_1} < \frac{13}{3}$, and if, moreover, $g \leq 23$, then $s(D) = \frac{a}{b_0}$ [FP05, Theorem 1.1(c) and Corollary 1.2]. In particular, for $g \leq 23$, $\overline{\mathcal{M}}_g$ is of general type if and only if there is an effective divisor D with $\frac{a}{b_0} < \frac{13}{2}$.

In their study of the Kodaira dimension of $\overline{\mathcal{M}}_g$ for $g \geq 24$, Harris, Mumford, and Eisenbud considered Brill-Noether divisors, defined as follows. For integers $r \geq 1$ and $d \leq g + r - 2$ such that

$$\rho(g, r, d) := g - (r + 1)(g - d + r) = -1, \tag{3}$$

one defines $\mathcal{M}^r_{g,d}$ to be the (divisorial part of the) locus of curves $[X] \in \mathcal{M}_g$ with a linear series $L \in W^r_d(X)$. The fact that this locus is not all of $\mathcal{M}_g$ is the essential content of the Brill-Noether Theorem [GH80]. Since the slope of the closure $\overline{\mathcal{M}}^r_{g,d}$ of $\mathcal{M}^r_{g,d}$ inside $\overline{\mathcal{M}}_g$ is $6 + \frac{12}{g+1}$ [HM82, EH87], it follows that $\overline{\mathcal{M}}_g$ is of general type when $g \geq 24$ and $g + 1$ is composite. (If $g + 1$ is prime, then the equation (3) has no solutions, and there is no Brill-Noether divisor on $\overline{\mathcal{M}}_g$.)

When g is even and at least 28, one similarly obtains a virtual divisor satisfying (2) supported on the closure of the locus of curves $[X] \in \mathcal{M}_g$ with a line bundle $L \in W^r_d(X)$ where $\rho(g, r, d) = 0$, such that the Petri map is not injective [EH87]. The fact that this locus is not all of $\overline{\mathcal{M}}_g$, from which it follows that this virtual class is effective, is the essential content of the Gieseker-Petri Theorem [Gie82]. Note that both the Brill-Noether Theorem and the Gieseker-Petri Theorem have more recent proofs by tropical arguments on chains of loops [CDPR12, JP14]. For $g < 24$ the Brill-Noether and Gieseker-Petri divisors do not satisfy inequality (2).

As explained in the introduction, for $g = 22$ and 23, we construct a different virtual divisor with smaller slope. Specifically, we consider the push forward of the virtual class of the locus of curves admitting a map of minimal degree to $\mathbf{P}^6$ with image contained in a quadric. Theorem 1.2 says that these virtual divisors satisfy (2), and Theorems 1.3 and 1.4 combine to show that they are effective.

2.2. **Test curves and intersection numbers.** We introduce a few standard test curves in $\overline{\mathcal{M}}_g$ that will be used several times in the paper. Choose a general pointed curve $[X, q]$ of genus $g - 1$. Then construct the families of stable curves of genus g

$$F_0 := \Big\{[X_{yq}] := [X/y \sim q] : y \in C\Big\} \subseteq \Delta_0 \subseteq \overline{\mathcal{M}}_g \quad \text{and} \tag{4}$$

$$F_{\text{ell}} := \Big\{[X \cup_q E_t] : t \in \mathbf{P}^1\Big\} \subseteq \Delta_1 \subseteq \overline{\mathcal{M}}_g, \tag{5}$$

where $\{[E_t, q]\}_{t\in \mathbf{P}^1}$ denotes a pencil of plane cubics and q is a fixed point of the pencil. The intersection of these test curves with the generators of $CH^1(\overline{\mathcal{M}}_g)$ is well-known, see [HM82]:

$$F_0\cdot\lambda=0,\qquad F_0\cdot\delta_0=2-2g,\qquad F_0\cdot\delta_1=1\ \text{ and }\qquad F_0\cdot\delta_j=0\ \text{ for }\ j=2,\ldots,\left\lfloor\frac{g}{2}\right\rfloor,$$
$$F_{\mathrm{ell}}\cdot\lambda=1,\qquad F_{\mathrm{ell}}\cdot\delta_0=12,\qquad F_{\mathrm{ell}}\cdot\delta_1=-1\ \text{ and }\qquad F_{\mathrm{ell}}\cdot\delta_j=0\ \text{ for } j=2,\ldots,\left\lfloor\frac{g}{2}\right\rfloor.$$

For a fixed pointed curve $[C,p]\in\mathcal{M}_{g-i,1}$ and a fixed curve $[D]\in\mathcal{M}_i$, we consider the family

$$F_i:=\Big\{[C\cup_{p\sim y}D]:y\in D\Big\}\subseteq\Delta_i\subseteq\overline{\mathcal{M}}_g. \tag{6}$$

Then using again [HM82] we find

$$F_i\cdot\lambda=0,\ F_i\cdot\delta_i=2-2i,\ \text{ and } F_i\cdot\delta_j=0\ \text{ for } j\neq i.$$

2.3. **A vanishing condition for effective divisors.** In order to prove Theorem 1.4, we must show that there is no nonzero effective divisor in $\overline{\mathcal{M}}_g$ over which the fibers of the degeneracy locus $\mathfrak{U}$ inside $\widetilde{\mathfrak{G}}^r_d$ are generically infinite. We will do so by applying certain sufficient vanishing conditions for effective divisors on $\overline{\mathcal{M}}_g$, which we explain next.

Definition 2.1. For $1\leq i,j<g$, let $\Delta_{i,j}\subseteq\overline{\mathcal{M}}_g$ be the codimension 2 boundary stratum parametrizing curves with two separating nodes, whose two tail components have genus i and j. Such a curve has a third component, of genus $g-i-j$, meeting each tail component at a node.

For $1<k\leq g$, let $\jmath_k\colon\overline{\mathcal{M}}_{k,1}\to\overline{\mathcal{M}}_g$ be the map obtained by attaching a fixed, general pointed curve $[X,p]\in\mathcal{M}_{g-k,1}$ to an arbitrary pointed curve of genus k. We let $\psi\in CH^1(\overline{\mathcal{M}}_{k,1})$ be the cotangent class, and we let $\delta_i=\delta_{i:1}\in CH^1(\overline{\mathcal{M}}_{k,1})$ denote the class of the closure of the locus of the union of two smooth curves of genera i and $k-i$, with the marked point lying on the genus i component, for $i=1,\ldots,k-1$. We then have the following formulas:

$$\jmath_k^*(\lambda)=\lambda,\qquad \jmath_k^*(\delta_0)=\delta_0,\qquad \jmath_k^*(\delta_k)=-\psi+\delta_{2k-g},\qquad \jmath_k^*(\delta_i)=\delta_{i+k-g}+\delta_{k-i}\ \text{for } i\neq k. \tag{7}$$

Here we make the convention $\delta_i:=0$, for $i<0$ or $i\geq g$.

The *Weierstrass divisor* $\overline{\mathcal{W}}_k$ in $\overline{\mathcal{M}}_{k,1}$ is the closure of the locus of smooth pointed curves $[X,p]$, where $p\in X$ is a Weierstrass point. The *hyperelliptic divisor* $\overline{\mathcal{H}}_3$ in $\overline{\mathcal{M}}_{3,1}$ is the locus where the underlying curve is hyperelliptic.

The following result provides sufficient intersection-theoretic conditions for an effective divisor on $\overline{\mathcal{M}}_g$ to be zero. It relies on the existing detailed knowledge of the Picard group of $\overline{\mathcal{M}}_{g,1}$.

Proposition 2.2. *Let $g\geq 6$ and let D be an effective divisor on $\overline{\mathcal{M}}_g$ with the following properties:*
1. *D is the closure of a divisor in $\mathcal{M}_g$;*
2. *$\jmath_2^*(D)=0$;*
3. *D does not contain any codimension 2 stratum $\Delta_{2,j}$.*
4. *if g is even then $\jmath_3^*(D)$ is a nonnegative combination of the classes $[\overline{\mathcal{W}}_3]$ and $[\overline{\mathcal{H}}_3]$ on $\overline{\mathcal{M}}_{3,1}$.*

Then $D=0$.

Proof. Write the class of D as

$$[D]=a\lambda-b_0\delta_0-\cdots-b_{\lfloor\frac{g}{2}\rfloor}\delta_{\lfloor\frac{g}{2}\rfloor}\in CH^1(\overline{\mathcal{M}}_g).$$

Since $\overline{\mathcal{M}}_g$ is projective, to show that D is zero, it suffices to show that $[D]=0$.

First, note that $a\geq 0$, since λ is nef and the complete curves disjoint from the boundary are dense in $\mathcal{M}_g$. Next, we claim that $b_i\geq 0$ for all i. For $i=2,\ldots,g-1$, the curve F_i moves in a family that covers the boundary divisor Δ_i. Since $\Delta_i\not\subseteq\operatorname{supp}(D)$, it follows that $F_i\cdot D\geq 0$, hence $b_i\geq 0$. For $i=0$, we similarly use the curve F_0 which moves in a family that covers Δ_0, to deduce that $(2g-2)b_0-b_1=F_0\cdot D\geq 0$, so $b_0\geq 0$.

By [EH87, Theorem 2.1], the condition $\jmath_2^*(D) = 0$ implies that $a = 10b_0 = 5b_1$ and $b_2 = 0$. Indeed, recall the relation $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$ in genus 2. By (7) we have $\jmath_2^*([D]) = a\lambda - b_0\delta_0 - b_1\delta_1 + b_2\psi$, and the conclusion follows from the fact that the classes $\psi, \delta_0, \delta_1 \in CH^1(\overline{\mathcal{M}}_{2,1})$ are independent.

Next we consider the test curve $F_{2,g-2-j} \subseteq \Delta_{2,g-j-2}$ obtained by gluing a fixed pointed curve of genus 2 to a moving point on a curve of genus j, which is itself glued at a fixed point to a curve of genus $g-j-2$. If $j \neq 2, g-4$, we have

$$F_{2,g-2-j} \cdot \lambda = 0, \qquad F_{2,g-2-j} \cdot \delta_2 = 1-2j, \qquad F_{2,g-2-j} \cdot \delta_j = 1,$$
$$F_{2,g-2-j} \cdot \delta_{j+2} = -1, \qquad F_{2,g-2-j} \cdot \delta_i = 0, \text{ for } i \neq 2, j, j+2.$$

Similarly, for the test curves $F_{2,g-4}$ and $F_{2,2}$ we have

$$F_{2,g-4} \cdot \lambda = 0, \qquad F_{2,g-4} \cdot \delta_i = 0, \text{ for } i \neq 2,4, \qquad F_{2,g-4} \cdot \delta_2 = -2, \qquad F_{2,g-4} \cdot \delta_4 = -1.$$
$$F_{2,2} \cdot \lambda = 0, \qquad F_{2,2} \cdot \delta_i = 0 \text{ for } i \neq 2,4, \qquad F_{2,2} \cdot \delta_2 = -4, \qquad F_{2,2} \cdot \delta_4 = 1.$$

Since $F_{2,g-2-j}$ covers the stratum $\Delta_{2,g-2-j}$, which is not contained in D, we have $F_{2,j} \cdot D \geq 0$. Since $b_2 = 0$, it follows that $b_j \leq b_{j+2}$ for $1 \leq j \leq g-3$, where we adopt the usual convention that $b_k := b_{g-k}$ for $\lfloor \frac{g}{2} \rfloor < k < g$. Replacing j by $g-j-2$, we see that

$$b_j = b_{j+2}, \text{ for } 1 \leq j \leq g-3. \tag{8}$$

When g is odd, combining (8) with the fact that $b_2 = 0$ shows that $b_j = 0$ for all $j \geq 1$. Since $a = 10b_0 = 5b_1$, it follows that $D = 0$.

It remains to consider the case when g is even. So far, using (8) and the relation $a = 10b_0 = 5b_1$, we have shown that

$$[D] = c\big(10\lambda - \delta_0 - 2\delta_1 - 2\delta_3 - \cdots - 2\delta_{\frac{g}{2}}\big)$$

for some $c \in \mathbb{Q}_{\geq 0}$. We aim to prove that $c = 0$.

By assumption, we have a relation $[\jmath_3^*(D)] = \alpha[\overline{\mathcal{W}}_3] + \beta[\overline{\mathcal{H}}_3]$, for certain nonnegative rational constants α and β. By [Cuk89] or [EH87], the class of the Weierstrass divisor is

$$[\overline{\mathcal{W}}_3] = -\lambda + \psi - 3\delta_1 - 6\delta_2.$$

By for instance [HM98, Section 3.H], the class of the hyperelliptic divisor is

$$[\overline{\mathcal{H}}_3] = 9\lambda - \delta_0 - 3\delta_1 - 3\delta_2.$$

Applying once more the formula (7), we find that

$$[\jmath_3^*(D)] = c\big(10\lambda - \delta_0 - 2\delta_2 + 2\psi\big) = \alpha\big(-\lambda - 3\delta_1 - 6\delta_2 + \psi\big) + \beta\big(9\lambda - \delta_0 - 3\delta_1 - 3\delta_2\big).$$

Since the classes $\lambda, \delta_0, \delta_1, \delta_2, \psi$ freely generate $CH^1(\overline{\mathcal{M}}_{3,1})$, we immediately obtain from this relation that $c = \alpha = \beta = 0$. That is, $D = 0$. □

2.4. **Tropical and nonarchimedean geometry of curves.** The techniques that we use to prove Theorems 1.3 and 1.4, and thereby establish the transversality statements needed to produce effective divisors of small slope on $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$, are based on tropical and nonarchimedean geometry. Let X be a curve of positive genus over an algebraically closed nonarchimedean field K with valuation ring R and residue field κ, of characteristic zero. For simplicity, we assume that K is spherically complete with value group $\mathbb{R}$. An example of such a field is $\mathbb{C}(\!(t^{\mathbb{R}})\!)$, the field of power series with real exponents and well-ordered support. See [Poo93] for an exposition of nonarchimedean fields with such completeness properties. Note that any two uncountable algebraically closed fields of the same cardinality and characteristic are isomorphic [Mar02, Proposition 2.2.5]. In particular, we may choose K to be isomorphic to $\mathbb{C}$, as an abstract field. The additional nonarchimedean structure on K gives us access to techniques from tropical geometry and Berkovich theory, just as the Euclidean norm on $\mathbb{C}$ gives access to techniques from Riemann surfaces and complex analytic geometry.

Since K is spherically complete with value group $\mathbb{R}$, every point in the nonarchimedean analytification X^{an} has type 1 or 2. Here, a type 1 point is simply a K-rational point, and a type 2 point

v corresponds to a valuation val_v on the function field $K(X)$ whose associated residue field is a transcendence degree 1 extension of the residue field κ of K. See [BPR13, BPR16] for details on the structure theory of curves over nonarchimedean fields, relations to tropical geometry, and proofs of the basic properties of analytification and tropicalization that we omit.

2.4.1. *Skeletons.* The *minimal skeleton* of X^{an} is the set of points with no neighborhood isomorphic to an analytic ball, and carries canonically the structure of a finite metric graph. More generally, a *skeleton* for X^{an} is the underlying set of a finite connected subgraph of X^{an} that contains this minimal skeleton. Any skeleton Γ is contained in the set of type 2 points, and any decomposition of a skeleton Γ into vertices and edges determines a semistable model of X over R. The vertices correspond to the irreducible components of the special fiber, and the irreducible component X_v corresponding to v has function field $\kappa(X_v)$, the transcendence degree 1 extension of κ given by the completion of $K(X)$ with respect to val_v. The edges correspond to the nodes of the special fiber, with the length of each edge given by the thickness of the corresponding node.

2.4.2. *Tropicalizations and reductions of rational functions.* Let Γ be a skeleton for X^{an}. Since Γ is contained in the set of type 2 points, for each nonzero rational function $f \in K(X)^*$ we get a real-valued function

$$\mathrm{trop}(f) : \Gamma \to \mathbb{R}, \quad v \mapsto \mathrm{val}_v(f).$$

This function is piecewise linear with integer slopes, and its slope along an edge incident to v is related to the reduction of f at v. This relation is known as the slope formula, a nonarchimedean analogue of the Poincaré-Lelong formula, which we now describe.

Given a nonzero rational function $f \in K(X)$ and a type 2 point $v \in X^{\mathrm{an}}$, choose $c \in K^*$ whose valuation is equal to $\mathrm{val}_v(f)$. Then f/c has valuation zero, and the *reduction* of f at v, denoted f_v, is defined to be its image in $\kappa(X_v)^*/\kappa^*$. This does not depend on the choice of c, so f_v is well-defined. Divisors of rational functions are invariant under multiplication by nonzero scalars, and we denote the divisor on X_v of any representative of f_v in $\kappa(X_v)^*$ by $\mathrm{div}(f_v)$. Each germ of an edge of Γ incident to v corresponds to a point of X_v (a node in the special fiber of a semistable model with skeleton Γ, in which X_v appears as a component). The slope formula then says that the outgoing slope of $\mathrm{trop}(f)$ along this germ of an edge is equal to the order of vanishing of f_v at that point [BPR13, Theorem 5.15(3)].

2.4.3. *Complete linear series on graphs.* Let $\mathrm{PL}(\Gamma)$ be the set of continuous piecewise linear functions on Γ with integer slopes. Throughout, we will use both the additive group structure on $\mathrm{PL}(\Gamma)$, and the tropical module structure given by pointwise minimum and addition of real scalars.

Given $v \in \Gamma$ and $\varphi \in \mathrm{PL}(\Gamma)$, the order of φ at v, denoted $\mathrm{ord}_v(\varphi)$, is the sum of the incoming slopes of φ at v. The principal divisor associated to φ is then $\mathrm{div}(\varphi) := \sum_{v\in\Gamma} \mathrm{ord}_v(\varphi)v$. The *complete linear series* of a divisor D on Γ is

$$R(D) := \big\{\varphi \in \mathrm{PL}(\Gamma) : \mathrm{div}(\varphi) + D \geq 0\big\}.$$

Note that $R(D) \subseteq \mathrm{PL}(\Gamma)$ is a tropical submodule, that is, it is closed under scalar addition and pointwise minimum.

By the Poincaré-Lelong formula, if D_X is any divisor on X tropicalizing to D and f is a section of $\mathcal{O}(D_X)$, then $\mathrm{trop}(f) \in R(D)$. We refer the reader to [BN07, Bak08] for further background on the divisor theory of graphs and metric graphs, and specialization from curves to graphs.

2.5. **Tropical independence.** We now recall the notion of tropical independence, as defined in [JP14], and prove Theorem 1.6. Let $\{\psi_i : i \in I\}$ be a finite collection of piecewise linear functions. A *tropical linear combination* is an expression

$$\theta = \min\{\psi_i + c_i\},$$

for some choice of real coefficients $\{c_i\}$. Note that different choices of coefficients may yield the same pointwise minimum, but we consider the coefficients $\{c_i\}$ to be part of the data in a tropical linear combination, so the tropical linear combinations of $\{\psi_i\}$ are naturally identified with $\mathbb{R}^I$.

Given a tropical linear combination $\theta = \min\{\psi_i + c_i\}$, we say that ψ_i *achieves the minimum* at $v \in \Gamma$ if $\theta(v) = \psi_i(v) + c_i$, and *achieves the minimum uniquely* if, moreover, $\theta(v) \neq \psi_j(v) + c_j$ for $j \neq i$. We say that *the minimum is achieved at least twice* at $v \in \Gamma$ if there are at least two distinct indices $i \neq j$ such that $\theta(v)$ is equal to both $\psi_i(v) + c_i$ and $\psi_j(v) + c_j$.

A *tropical dependence* is a tropical linear combination $\theta = \min\{\psi_i + c_i\}$ such that the minimum of the functions $\psi_i + c_i$ is achieved at least twice at every point of Γ. Equivalently, $\theta = \min\{\psi_i + c_i\}$ is a tropical dependence if $\theta = \min_{j \neq i}\{\psi_j + c_j\}$, for all i. If no such tropical linear combination exists, then $\{\psi_0, \dots \psi_r\}$ is *tropically independent.*

Most importantly for applications to Brill-Noether theory, if a set of nonzero functions $\{f_0, \dots, f_r\}$ is linearly dependent over K, then the set of tropicalizations $\{\mathrm{trop}(f_0), \dots, \mathrm{trop}(f_r)\}$ is tropically dependent [JP14, Lemma 3.2]. Therefore, tropical independence of the tropicalizations is a sufficient condition for linear independence of rational functions.

Our arguments in this paper use the following new characterization of tropical independence.

Definition 2.3. A tropical linear combination $\theta = \min\{\psi_i + c_i\}$ is an *independence* if each ψ_i achieves the minimum uniquely at some point $v \in \Gamma$.

Equivalently, $\theta = \min\{\psi_i + c_i\}$ is an independence if $\theta \neq \min_{j \neq i}\{\psi_j + c_j\}$, for all $i \in I$.

Remark 2.4. In linear algebra, a *dependence* is a linear combination that shows a collection of vectors is linearly dependent. Similarly, a tropical dependence is a tropical linear combination that shows a collection of PL functions is tropically dependent. The word *independence* is chosen to parallel this established terminology; by Theorem 1.6, the existence of an independence among a collection of PL functions shows that these functions are tropically independent. There is no analogous notion in linear algebra.

Recall that Theorem 1.6 says a finite subset $\{\psi_i : i \in I\} \subseteq \mathrm{PL}(\Gamma)$ is tropically independent if and only if there is an independence $\theta = \min\{\psi_i + c_i\}$.

Proof of Theorem 1.6. First, we suppose that $\{\psi_i\}$ is tropically dependent, and show that there is no such independence. Choose real coefficients c'_i such that the minimum of $\{\psi_i + c'_i\}$ occurs at least twice at every point $v \in \Gamma$. Now, consider an arbitrary tropical linear combination $\theta = \min\{\psi_i + c_i\}$. Choose $j \in I$ so that $c_j - c'_j$ is maximal. Then $\psi_j + c_j \geq \min_{j \neq i}\{\psi_i + c_i\}$ at every point $v \in \Gamma$, and hence $\theta = \min\{\psi_i + c_i\}$ is not an independence.

It remains to show that if there is no such independence, then $\{\psi_i\}$ is tropically dependent. Let A_i be the set of vectors $c = (c_0, \dots, c_r) \in \mathbb{R}^{r+1}$ such that $\psi_i(v) + c_i \geq \min_{j \neq i}\{\psi_j(v) + c_j\}$ for all $v \in \Gamma$. Note that c gives an independence if and only if it is contained in none of the A_i. Similarly, c gives a tropical dependence if and only if it is contained in all of the A_i. Hence, we must show that if the sets A_i cover $\mathbb{R}^{r+1}$ then their intersection is nonempty.

Suppose $A_0 \cup \cdots \cup A_r = \mathbb{R}^{r+1}$. Choose m sufficiently large so that $\psi_i(v) + m > \psi_j(v)$ for all i, j and all $v \in \Gamma$. Let Δ be the simplex spanned by mr times the standard basis vectors in $\mathbb{R}^{r+1}$. Note that the face spanned by the vertices corresponding to any subset $I \subseteq \{0, \dots, r\}$ is covered by $\{A_i \cap \Delta\}_{i \in I}$. The Knaster-Kuratowski-Mazurkiewicz lemma (that is, the set-covering variant of the Brouwer fixed-point theorem) then says that $A_0 \cap \cdots \cap A_r \cap \Delta$ is nonempty, as required. □

3. Constructing the virtual divisors

In this section, we construct the virtual divisor class $[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}}$ as the push forward of the virtual class of a codimension 3 determinantal locus. This determinantal locus is contained inside a universal parameter space of limit linear series of type $\mathfrak{g}^6_{26}$ over an open substack $\widetilde{\mathfrak{M}}_{23}$ of $\overline{\mathfrak{M}}_{23}$ that differs from $\mathfrak{M}_{23} \cup \Delta_0 \cup \Delta_1$ outside a subset of codimension 2. We follow a similar procedure in the case of the virtual divisor classes $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$ on $\widetilde{\mathfrak{M}}_g$, for $g = 2s^2+s+1$, with $s \geq 2$. As long as the two constructions run parallel, we treat both simultaneously. Throughout, we work over an algebraically closed field K of characteristic zero.

We first recall the notation for vanishing and ramification sequences of limit linear series [EH86].

Definition 3.1. Let X be a smooth curve of genus g, $q \in X$ a point, and $\ell = (L, V) \in G^r_d(X)$ a linear series on X. The *ramification sequence* of ℓ at q

$$\alpha^\ell(q) : \alpha^\ell_0(q) \leq \cdots \leq \alpha^\ell_r(q)$$

is obtained from the *vanishing sequence*

$$a^\ell(q) : a^\ell_0(q) < \cdots < a^\ell_r(q) \leq d$$

by setting $\alpha^\ell_i(q) := a^\ell_i(q) - i$, for $i = 0, \ldots, r$. Sometimes, when L is clear from the context, we write $\alpha^V(q) = \alpha^\ell(q)$ and similarly $a^V(q) = a^\ell(q)$. The *ramification weight* of q with respect to ℓ is $\mathrm{wt}^\ell(q) := \sum_{i=0}^r \alpha^\ell_i(q)$. We denote by $\rho(\ell, q) := \rho(g, r, d) - \mathrm{wt}^\ell(q)$ the *adjusted Brill-Noether number* of ℓ with respect to q.

Recall from [EH87, p. 364] that a *generalized limit linear series* on a tree-like curve X consists of a collection $\ell = \{(L_C, V_C) : C \text{ is a component of } X\}$, where L_C is a rank 1 torsion free sheaf of degree d on C and $V_C \subseteq H^0(C, L_C)$ is an $(r+1)$-dimensional space of sections satisfying the usual compatibility condition on the vanishing sequences at the nodes of X. For such a tree-like curve X, we denote by $\overline{G}^r_d(X)$ the variety of generalized limit linear series of type $\mathfrak{g}^r_d$.

In what follows we fix positive integers g, r, and d such that either

$$g = 23, \ r = 6, \ d = 26, \quad \text{or} \tag{9}$$

$$g = 2s^2+s+1, \ r = 2s, \ d = 2s^2+2s+1, \quad \text{where} \quad s \geq 2. \tag{10}$$

Note that $\rho(g, r, d) = 2$ in case (9) and $\rho(g, r, d) = 1$ in case (10).

3.1. **An open substack of $\overline{\mathfrak{M}}_g$.** We denote by $\mathcal{M}^r_{g,d-1}$ the closed subvariety of $\mathcal{M}_g$ parametrizing curves X such that $W^r_{d-1}(X) \neq \emptyset$. We claim that $\mathrm{codim}(\mathcal{M}^r_{g,d-1}, \mathcal{M}_g) \geq 2$. To see this, it suffices to observe that $\rho(g, r, d-1)$ is less than -1, and then apply [EH89, Theorem 1.1]. For each curve $[X] \in \mathcal{M}_g \setminus \mathcal{M}^r_{g,d-1}$, every line bundle $L \in W^r_d(X)$ is base point free, with $H^1(X, L^{\otimes 2}) = 0$, since $d \geq g$. We denote by $\overline{\mathcal{M}}^r_{g,d-1}$ the closure of $\mathcal{M}^r_{g,d-1}$ in $\overline{\mathcal{M}}_g$.

Let $\Delta_1^\circ \subseteq \Delta_1 \subseteq \overline{\mathcal{M}}_g$ be the locus of curves $[X \cup_y E]$, where X is a smooth curve of genus $g-1$ and $[E, y] \in \overline{\mathcal{M}}_{1,1}$ is an arbitrary elliptic curve. The point of attachment $y \in X$ is chosen arbitrarily. Furthermore, let $\Delta_0^\circ \subseteq \Delta_0 \subseteq \overline{\mathcal{M}}_g$ be the locus of curves $[X_{yq} := X/y \sim q] \in \Delta_0$, where $[X, q]$ is a smooth curve of genus $g-1$ and $y \in X$ is an arbitrary point, together with their degenerations $[X \cup_q E_\infty]$, where E_∞ is a rational nodal curve (that is, E_∞ is a nodal elliptic curve and $j(E_\infty) = \infty$). Points of this form comprise the intersection $\Delta_0^\circ \cap \Delta_1^\circ$. We define the following open subset of $\overline{\mathcal{M}}_g$:

$$\overline{\mathcal{M}}_g^\circ := \mathcal{M}_g \cup \Delta_0^\circ \cup \Delta_1^\circ.$$

In order to define the open substack of $\overline{\mathcal{M}}_g^\circ$ over which Theorems 1.2 and 1.7 will be ultimately proved, we need further notation. Let $\mathcal{T}_0$ be the subvariety of Δ_0° of curves $[X_{yq} := X/y \sim q]$, where the curve X satisfies $\overline{G}^{r+1}_d(X) \neq \emptyset$ or $\overline{G}^r_{d-2}(X) \neq \emptyset$. Similarly, $\mathcal{T}_1 \subseteq \Delta_1^\circ$ denotes the subvariety of curves $[X \cup_y E]$, where X is a smooth curve of genus $g-1$ with $G^{r+1}_d(X) \neq \emptyset$ or $G^r_{d-2}(X) \neq \emptyset$. Observe that both $\mathcal{T}_0$ and $\mathcal{T}_1$ are closed in $\overline{\mathcal{M}}_g^\circ$.

We introduce the following open subset of $\overline{\mathcal{M}}_g$:

$$\widetilde{\mathcal{M}}_g := \overline{\mathcal{M}}_g^\circ \setminus \Big(\overline{\mathcal{M}}^r_{g,d-1} \cup \mathcal{T}_0 \cup \mathcal{T}_1\Big). \tag{11}$$

We define $\widetilde{\Delta}_0 := \widetilde{\mathcal{M}}_g \cap \Delta_0 \subseteq \Delta_0^\circ$ and $\widetilde{\Delta}_1 := \widetilde{\mathcal{M}}_g \cap \Delta_1 \subseteq \Delta_1^\circ$, so

$$\widetilde{\mathcal{M}}_g = \big(\mathcal{M}_g \setminus \mathcal{M}^r_{g,d-1}\big) \cup \widetilde{\Delta}_0 \cup \widetilde{\Delta}_1.$$

Note that $\widetilde{\mathcal{M}}_g$ and $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$ differ outside a set of codimension 2 and we use the identification $\mathrm{Pic}(\widetilde{\mathcal{M}}_g) \cong CH^1(\widetilde{\mathcal{M}}_g) = \mathbb{Q}\langle \lambda, \delta_0, \delta_1\rangle$, where λ is the Hodge class, $\delta_0 := [\widetilde{\Delta}_0]$ and $\delta_1 := [\widetilde{\Delta}_1]$.

3.2. **Stacks of limit linear series.** Next we introduce the parameter spaces of limit linear series that we will use.

Definition 3.2. Let $\widetilde{\mathfrak{G}}^r_d$ be the stack of pairs $[X, \ell]$, where $[X] \in \widetilde{\mathfrak{M}}_g$ and ℓ is a (generalized) limit linear series on the tree-like curve X in the sense of [EH87]. We consider the proper projection map

$$\sigma \colon \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathfrak{M}}_g.$$

We refer to [EH86] and [EH87] for facts on limit linear series and to [Oss06] and [LO19] for details regarding the construction of $\widetilde{\mathfrak{G}}^r_d$. We discuss the fibers of σ. Over a curve $[X \cup_y E] \in \widetilde{\Delta}_1$, we identify $\sigma^{-1}([X \cup_y E])$ with the variety of limit linear series $\ell = (\ell_X, \ell_E) \in G^r_d(X) \times G^r_d(E)$ satisfying the compatibility conditions described in [EH86]. Over a point $[X \cup_y E_\infty] \in \widetilde{\Delta}_0 \cap \widetilde{\Delta}_1$, the fiber $\sigma^{-1}([X \cup_y E_\infty])$ is identified with the variety of generalized limit linear series $\overline{G}^r_d(X \cup_y E_\infty)$. In order to describe the fiber $\sigma^{-1}([X_{yq}])$ over an irreducible curve $[X_{yq}] \in \widetilde{\Delta}_0$, we recall a few things about the variety $\overline{W}^r_d(X_{yq})$ of rank 1 torsion free sheaves L on X_{yq} having $\deg(L) = d$ and $h^0(X_{yq}, L) \geq r+1$. We denote by $W^r_d(X_{yq})$ the open subvariety of $\overline{W}^r_d(X_{yq})$ consisting of line bundles. If $\nu \colon X \to X_{yq}$ is the normalization map, and the curve X satisfies $\overline{G}^{r+1}_d(X) = \emptyset$, we observe that $h^0(X_{yq}, L) = r+1$ for every sheaf $L \in \overline{W}^r_d(X_{yq})$. In particular, we identify the fiber $\sigma^{-1}([X_{yq}])$ with $\overline{W}^r_d(X_{yq})$. Moreover, the pull back map $\nu^* \colon W^r_d(X_{yq}) \to \mathrm{Pic}^d(X)$ is injective.

For a pointed curve $[X, y, q] \in \mathcal{M}_{g-1,2}$, by [OS79, Proposition 12.1], there is a desingularization of the compactified Jacobian

$$\widetilde{\mathrm{Pic}}^d(X_{yq}) := \mathbf{P}\big(\mathcal{P}_y \oplus \mathcal{P}_q\big) \to \overline{\mathrm{Pic}}^d(X_{yq}).$$

Here, $\mathcal{P}$ denotes a Poincaré bundle on $X \times \mathrm{Pic}^d(X)$, $\mathcal{P}_y$ denotes the restriction of $\mathcal{P}$ to $\{y\} \times \mathrm{Pic}^d(X)$, and $\mathcal{P}_q$ denotes the restriction of $\mathcal{P}$ to $\{q\} \times \mathrm{Pic}^d(X)$. A point in $\widetilde{\mathrm{Pic}}^d(X_{yq})$ can be thought of as a pair (L, Q), where L is a line bundle of degree d on X and $L_y \oplus L_q \twoheadrightarrow Q$ is a 1-dimensional quotient. The map $\widetilde{\mathrm{Pic}}^d(X_{yq}) \to \overline{\mathrm{Pic}}^d(X_{yq})$ assigns to a pair (L, Q) the sheaf L' on X_{yq}, defined by the exact sequence

$$0 \longrightarrow L' \longrightarrow \nu_* L \longrightarrow Q \longrightarrow 0.$$

Remark 3.3. If the rank 1 torsion free sheaf $L \in \overline{W}^r_d(X_{yq}) \setminus W^r_d(X_{yq})$ is not locally free, then this point corresponds to *two* points in $\widetilde{\mathrm{Pic}}^d(X_{yq})$. If $A \in W^r_{d-1}(X)$ is the unique line bundle such that $\nu_*(A) = L$, then these points are $\big(A(q) = A \otimes \mathcal{O}_X(q), A(q)_q\big)$ and $\big(A(y) = A \otimes \mathcal{O}_X(y), A(y)_y\big)$ respectively.

Let $\widetilde{\mathfrak{C}}_g \to \widetilde{\mathfrak{M}}_g$ be the universal curve, and let $p_2 \colon \widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathfrak{G}}^r_d$ be the projection map. We denote by $\mathfrak{Z} \subseteq \widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d$ the codimension 2 substack consisting of pairs $[X_{yq}, L, z]$, where $[X_{yq}] \in \Delta_0^\circ$, the point z is the node of X_{yq} and $L \in \overline{W}^r_d(X_{yq}) \setminus W^r_d(X_{yq})$ is a *non-locally free* torsion free sheaf. Let

$$\epsilon : \widehat{\mathfrak{C}}_g := \mathrm{Bl}_{\mathfrak{Z}}\Big(\widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d\Big) \to \widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d$$

be the blow-up of this locus, and we denote the induced universal curve by

$$\wp := p_2 \circ \epsilon : \widehat{\mathfrak{C}}_g \to \widetilde{\mathfrak{G}}^r_d.$$

The fiber of $\wp$ over a point $[X_{yq}, L] \in \widetilde{\Delta}_0$, where $L \in \overline{W}^r_d(X_{yq}) \setminus W^r_d(X_{yq})$, is the semistable curve $X \cup_{\{y,q\}} R$ of genus g, where R is a smooth rational curve meeting X transversally at y and q.

3.3. **A degeneracy locus in the universal linear series.** We choose a Poincaré line bundle $\mathcal{L}$ over $\widehat{\mathfrak{C}}_g$ having the following properties:

(1) For a curve $[X \cup_y E] \in \widetilde{\Delta}_1$ and a limit linear series $\ell = (\ell_X, \ell_E) \in G^r_d(X) \times G^r_d(E)$, we have that $\mathcal{L}_{|[X\cup_y E,\ell]} \in \mathrm{Pic}^d(X) \times \mathrm{Pic}^0(E)$, where the restriction $\mathcal{L}_{|E}$ is obtained by twisting the underlying line bundle L_E of the E-aspect ℓ_E by $\mathcal{O}_E(-dy)$.

(2) For a point $t = [X_{yq}, L]$, where $[X_{yq}] \in \widetilde{\Delta}_0$ and $L \in \overline{W}^r_d(X_{yq}) \setminus W^r_d(X_{yq})$, thus $L = \nu_*(A)$ for some $A \in W^r_{d-1}(X)$, we have $\mathcal{L}_{|X} \cong A$ and $\mathcal{L}_{|R} \cong \mathcal{O}_R(1)$. Here, as before, $\wp^{-1}(t) = X \cup R$.

Next we introduce the sheaves

$$\mathcal{E} := \wp_*(\mathcal{L}) \quad \text{and } \mathcal{F} := \wp_*(\mathcal{L}^{\otimes 2}) \tag{12}$$

which play an essential role in the paper. By Grauert's theorem, $\mathcal{E}$ is locally free and $\mathrm{rank}(\mathcal{E}) = r+1$, and $\mathcal{F}_{\widetilde{\mathfrak{G}}^r_d \setminus \sigma^{-1}(\widetilde{\Delta}_1)}$ is also locally free and $\mathrm{rank}(\mathcal{F}) = 2d+1-g$. We will show in Proposition 3.6 that in fact $\mathcal{F}$ is locally free over $\widetilde{\mathfrak{G}}^r_d$, and give a geometric interpretation of its fibers.

There is a natural vector bundle morphism over $\widetilde{\mathfrak{G}}^r_d$ given by multiplication of sections,

$$\phi\colon \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}. \tag{13}$$

We denote by $\mathfrak{U} \subseteq \widetilde{\mathfrak{G}}^r_d$ the first degeneracy locus of ϕ, which carries a natural virtual class in the expected codimension, as the next definition explains.

Definition 3.4. We define the virtual divisor class $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}} := \sigma_*([\mathfrak{U}]^{\mathrm{virt}})$. Precisely, the classes $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$ are virtual divisors in $\widetilde{\mathfrak{M}}_g$ given by

$$[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}} := \sigma_*\Big(c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\Big) \in CH^1(\widetilde{\mathfrak{M}}_{23}),$$

and, for $s \geq 2$ and $g = 2s^2+s+1$,

$$[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}} := \sigma_*\Big(c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\Big) \in CH^1(\widetilde{\mathfrak{M}}_g).$$

In order to establish the local freeness of $\mathcal{F}$ and understand better the morphism ϕ in (13), we need further preparation. For a pointed curve $[X, y] \in \mathcal{M}_{g-1,1}$, we denote by X'_y the genus g curve obtained from X by creating a cusp at y and by $\nu\colon X \to X'_y$ the normalization map. Recall that a *pseudo-stable* curve is a connected curve having only nodes and cusps as singularities, such that its dualizing sheaf is ample and each smooth irreducible component of genus 1 intersects the rest of the curve in at least two points. Pseudo-stable curves of genus g form a Delige-Mumford stack $\overline{\mathfrak{M}}^{\mathrm{ps}}_g$. One has a divisorial contraction $\pi\colon \overline{\mathfrak{M}}_g \to \overline{\mathfrak{M}}^{\mathrm{ps}}_g$ replacing each elliptic tail of a stable curve with a cusp [HH09]. Set-theoretically, $\pi\big([X \cup_y E]\big) = [X'_y]$.

Definition 3.5. Let $\mathcal{W} \subseteq \sigma^{-1}(\widetilde{\Delta}_1) \subseteq \widetilde{\mathfrak{G}}^r_d$ be the divisor consisting of pairs $[X \cup_y E, \ell]$, where $[X \cup_y E] \in \widetilde{\Delta}_1$ and $\ell = (\ell_X, \ell_E)$ is a limit linear series on $X \cup_y E$ such that $L_E = \mathcal{O}_E(dy)$.

If $[X \cup_y E, \ell] \in \sigma^{-1}(\widetilde{\Delta}_1)$, then the X-aspect ℓ_X of ℓ has a cusp at the point y. From the definition (11) of $\widetilde{\mathcal{M}}_g$, it follows that ℓ_X must be complete. Arguing along the lines of [HH09] one sees that there is a divisorial contraction $\widetilde{\pi}\colon \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathfrak{G}}^{r,\mathrm{ps}}_d$ of $\sigma^{-1}(\widetilde{\Delta}_1)$, where $\widetilde{\mathfrak{G}}^{r,\mathrm{ps}}_d$ denotes the stack of linear series of type $\mathfrak{g}^r_d$ over curves from the open substack $\pi(\widetilde{\mathfrak{M}}_g)$ of $\overline{\mathfrak{M}}^{\mathrm{ps}}_g$. The morphism $\widetilde{\pi}$ replaces each curve $[X \cup_y E] \in \widetilde{\Delta}_1$ with the cuspidal curve X'_y and a limit linear series $\big(\ell_X = |L_X|, \ell_E\big)$ on $X \cup_y E$ (where note that L_X is locally free) with the line bundle $L'_X \in W^r_d(X'_y)$ such that $\nu^*(L'_X) = L_X$. Observe that $h^0\big(X, L_X(-2y)\big) < h^0(X, L_X)$ since $[X \cup_y E] \notin \mathcal{T}_1$, therefore the line bundle L'_X on X'_y is uniquely determined by its pull back L_X under the normalization map $\nu\colon X \to X'_y$.

If we denote by Υ the divisor in $\wp^{-1}\big(\sigma^{-1}(\widetilde{\Delta}_1)\big)$ corresponding to marked points lying on the elliptic tail, then the morphism $\widetilde{\pi}:\widetilde{\mathfrak{G}}^r_d\to\widetilde{\mathfrak{G}}^{r,\mathrm{ps}}_d$ is induced by the linear series $\big|\wp_*\big(\omega_\wp(\Upsilon)\big)\big|$ (see [HH09, Proposition 3.8] for a very similar claim). We denote by

$$\widetilde{\wp}:\widehat{\mathfrak{C}}^{\mathrm{ps}}_g\to\widetilde{\mathfrak{G}}^{r,\mathrm{ps}}_d$$

the universal curve and by $\mathcal{L}^{\mathrm{ps}}$ the Poincaré bundle on $\widehat{\mathfrak{C}}^{\mathrm{ps}}_g$.

After this preparation, we now describe the morphism ϕ defined in (13) in more detail.

Proposition 3.6. *Both sheaves $\mathcal{E}=\wp_*(\mathcal{L})$ and $\mathcal{F}=\wp_*(\mathcal{L}^{\otimes 2})$ are locally free over $\widetilde{\mathfrak{G}}^r_d$.*

Proof. We first show that for any $t\in\widetilde{\mathfrak{G}}^r_d$, one has $h^0\big(\wp^{-1}(t),\mathcal{L}_{|\wp^{-1}(t)}\big)=r+1$. Since the claim obviously holds for points in $\sigma^{-1}(\mathcal{M}_g\setminus\mathcal{M}^r_{g,d-1})$, we assume first that $t=(X\cup_y E,\ell_X,\ell_E)\in\sigma^{-1}(\widetilde{\Delta}_1)$. Since $[X\cup_y E]\notin\mathcal{T}_1$, we have $h^0(X,L_X)=r+1$ and thus $\ell_X=|L_X|$ is a complete linear series. We have the exact sequence on $\wp^{-1}(t)$

$$0\longrightarrow H^0\big(\wp^{-1}(t),\mathcal{L}_{|\wp^{-1}(t)}\big)\longrightarrow H^0(X,L_X)\oplus H^0\big(E,L_E(-dy)\big)\xrightarrow{\mathrm{ev}_y}\mathcal{O}_y,\qquad(14)$$

where $\mathcal{O}_y$ is the structure sheaf of the point y. We distinguish two cases. If $L_E\ncong\mathcal{O}_E(dy)$, then $a^{\ell_E}_r(y)<d$, hence $a^{\ell_X}_0(y)>0$ and ℓ_X has a base point at y, in which case from (14) we get $H^0\big(\wp^{-1}(t),\mathcal{L}_{\wp^{-1}(t)}\big)\cong H^0(X,L_X)\cong H^0\big(X,L_X(-y)\big)$, which is $(r+1)$-dimensional.

If on the other hand $L_E\cong\mathcal{O}_E(dy)$, then by restricting to the second factor, we see that the evaluation map ev_y is surjective, and again from (14) we obtain that $h^0\big(\wp^{-1}(t),\mathcal{L}_{|\wp^{-1}(t)}\big)=r+1$.

Assume now that $t=[X_{yq},L]\in\sigma^{-1}(\widetilde{\Delta}_0)$. The case where L is locally free is clear. Assume instead that $L=\nu_*(A)$, with $A\in W^r_{d-1}(X)$. Recall that $\wp^{-1}(t)=X\cup_{\{y,q\}}R$, with R being a smooth rational curve meeting X at the points y and q. The Mayer-Vietoris sequence on $\wp^{-1}(t)$ then gives rise to exact sequences

$$0\longrightarrow H^0\big(\wp^{-1}(t),\mathcal{L}_{|\wp^{-1}(t)}\big)\longrightarrow H^0(X,A)\oplus H^0(R,\mathcal{O}_R(1))\longrightarrow\mathcal{O}_y\oplus\mathcal{O}_q,$$

and

$$0\longrightarrow H^0\big(\wp^{-1}(t),\mathcal{L}^{\otimes 2}_{|\wp^{-1}(t)}\big)\longrightarrow H^0(X,A^{\otimes 2})\oplus H^0(R,\mathcal{O}_R(2))\longrightarrow\mathcal{O}_y\oplus\mathcal{O}_q.$$

Since $[X_{yq}]\notin\mathcal{T}_0$, it follows that $h^0(X,A)=r+1$. Again, by restricting to the second factor, we see that the righthand map is surjective. Thus $h^0\big(\wp^{-1}(t),\mathcal{L}_{|\wp^{-1}(t)}\big)=r+1$ for every $t\in\widetilde{\mathfrak{G}}^r_d$, which shows that $\mathcal{E}$ is locally free.

We now turn our attention to the sheaf $\mathcal{F}$ and first show that for $t\in\widetilde{\mathfrak{G}}^r_d\setminus\mathcal{W}$ we have that

$$h^0\big(\wp^{-1}(t),\mathcal{L}^{\otimes 2}_{|\wp^{-1}(t)}\big)=2d+1-g$$

The case $t=[X_{yq},\nu_*(A)]\in\sigma^{-1}(\widetilde{\Delta}_0)$ follows from the second exact sequence above. Specifically, we have $h^1(X,A^{\otimes 2})=0$, so $h^0(X,A^{\otimes 2})=2d-g$, and by restricting to the second factor, we see that the righthand map is surjective.

If now $t=(X\cup_y E,\ell_X,\ell_E)\in\sigma^{-1}(\widetilde{\Delta}_1)$, we have an exact sequence

$$0\longrightarrow H^0\big(\wp^{-1}(t),\mathcal{L}^{\otimes 2}_{|\wp^{-1}(t)}\big)\longrightarrow H^0(X,L_X^{\otimes 2})\oplus H^0\big(E,L_E^{\otimes 2}(-2dy)\big)\xrightarrow{\mathrm{ev}_y}\mathcal{O}_y.$$

Since $h^1(X,L_X^{\otimes 2})=h^1(X,L_X^{\otimes 2}(-y))=0$, it follows that the map ev_y in the previous sequence is surjective. If $L_E\ncong\mathcal{O}_E(dy)$, we obtain $h^0\big(p^{-1}(t),\mathcal{L}^{\otimes 2}_{|p^{-1}(t)}\big)=2d+1-g$.

If $t\in\mathcal{W}$, then $L_E=\mathcal{O}_E(dy)$ and $h^0\big(\wp^{-1}(t),\mathcal{L}^{\otimes 2}_{|\wp^{-1}(t)}\big)=2d+2-g$ and this argument breaks down. Instead, we recall that we introduced the divisorial contraction $\widetilde{\pi}\colon\widetilde{\mathfrak{G}}^r_d\to\widetilde{\mathfrak{G}}^{r,\mathrm{ps}}_d$ of $\sigma^{-1}(\widetilde{\Delta}_1)$. Then

$$\wp_*(\mathcal{L}^{\otimes 2})=\widetilde{\pi}^*\Big(\widetilde{\wp}_*\big((\mathcal{L}^{\mathrm{ps}})^{\otimes 2}\big)\Big).$$

That is, for each $t \in \mathcal{W}$, the linear series $\wp_*(\mathcal{L}^{\otimes 2})_{|\wp^{-1}(t)}$ replaces the elliptic tail with a cusp. Since $h^0\big(X'_y, (L^{\mathrm{ps}})^{\otimes 2}\big) = 2d+1-g$ for every cuspidal curve X'_y and each $L^{\mathrm{ps}} \in W^r_d(X'_y)$, applying Grauert's theorem over $\widetilde{\mathfrak{G}}^{r,\mathrm{ps}}_d$, we conclude that the sheaf $\mathcal{F} = \wp(\mathcal{L}^{\otimes 2})$ is locally free as well. □

Remark 3.7. In situation (10) the local freeness of $\mathcal{F}$ follows from general principles, without having to resort to the local analysis above. Indeed, applying [Har80, Corollary 1.7], it follows that $\wp_*(\mathcal{L}^{\otimes 2})$ is a reflexive sheaf, thus its singular locus is of codimension at least 3 in $\widetilde{\mathfrak{G}}^r_d$. Removing this locus, one can still define the virtual class $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$ as in Definition 3.4. This argument falls short in case (9), however, where we cannot discard codimension 3 loci in $\widetilde{\mathfrak{G}}^r_d$.

The next corollary summarizes the fiberwise description of $\mathcal{E}$ and $\mathcal{F}$ implicitly obtained above. It follows from the application of Grauert's Theorem explained in the proof of Proposition 3.6.

Corollary 3.8. *The vector bundle map* $\phi\colon \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ *has the following local description:*
(i) For $[X, L] \in \widetilde{\mathfrak{G}}^r_d$*, with* $[X] \in \mathcal{M}_g \setminus \mathcal{M}^r_{g,d-1}$ *smooth, one has the following description of the fibers*

$$\mathcal{E}_{(X,L)} = H^0(X, L) \quad \textit{and} \quad \mathcal{F}_{(X,L)} = H^0(X, L^{\otimes 2})$$

and $\phi_{(X,L)} : \mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$ *is the usual multiplication map of global sections.*

(ii) Suppose $t = (X \cup_y E, \ell_X, \ell_E) \in \sigma^{-1}(\widetilde{\Delta}_1)$*, where* X *is a curve of genus* $g-1$*,* E *is an elliptic curve and* $\ell_X = |L_X|$ *is the* X*-aspect of the corresponding limit linear series with* $L_X \in W^r_d(X)$ *such that* $h^0(X, L_X(-2y)) \geq r$*. If* L_X *has no base point at* y*, then*

$$\mathcal{E}_t = H^0(X, L_X) \cong H^0\big(X, L_X(-2y)\big) \oplus K \cdot u \quad \textit{and} \quad \mathcal{F}_t = H^0\big(X, L_X^{\otimes 2}(-2y)\big) \oplus K \cdot u^2,$$

where $u \in H^0(X, L_X)$ *is any section such that* $\mathrm{ord}_y(u) = 0$*.*

If L_X *has a base point at* y*, then* $\mathcal{E}_t = H^0(X, L_X) \cong H^0(X, L_X(-y))$ *and the image of the map* $\mathcal{F}_t \to H^0(X, L_X^{\otimes 2})$ *is the subspace* $H^0\big(X, L_X^{\otimes 2}(-2y)\big) \subseteq H^0(X, L_X^{\otimes 2})$*.*

(iii) Let $t = [X_{yq}, L] \in \sigma^{-1}(\widetilde{\Delta}_0)$ *be a point with* $q, y \in X$ *and let* $L \in W^r_d(X_{yq})$ *be a locally free sheaf of rank* 1*, such that* $h^0(X, \nu^* L(-y-q)) \geq r$*, where* $\nu\colon X \to X_{yq}$ *is the normalization map. Then*

$$\mathcal{E}(t) = H^0(X, \nu^* L) \quad \textit{and} \quad \mathcal{F}(t) = H^0\big(X, \nu^* L^{\otimes 2}(-y-q)\big) \oplus K \cdot u^2,$$

where $u \in H^0(X, \nu^* L)$ *is any section not vanishing at both points* y *and* q*.*

(iv) Let $t = [X_{yq}, \nu_*(A)]$*, where* $A \in W^r_{d-1}(X)$ *and set again* $X \cup_{\{y,q\}} R$ *to be the fiber* $\wp^{-1}(t)$*. Then* $\mathcal{E}(t) = H^0(X \cup R, \mathcal{L}_{X \cup R}) \cong H^0(X, A)$ *and* $\mathcal{F}(t) = H^0(X \cup R, \mathcal{L}^{\otimes 2}_{X \cup R})$*. Furthermore,* $\phi(t)$ *is the multiplication map on* $X \cup R$*.*

3.4. **Pull back to test curves.** In preparation for the proofs of Theorems 1.2 and 1.7, concerning the calculation of $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$, we describe the restriction of the morphism ϕ along the pull backs of the three standard test curves F_0, F_{ell} and F_1 defined by (4), (5) and (6), respectively. Recall that we fix a general pointed curve $[X, q]$ of genus $g-1$ and a pointed elliptic curve $[E, y]$. We then have

$$F_0 := \Big\{X_{yq} := X/y \sim q : y \in X\Big\} \subseteq \Delta_0^\circ \subseteq \overline{\mathcal{M}}_g^\circ \ \text{ and } \ F_1 := \Big\{X \cup_y E : y \in X\Big\} \subseteq \Delta_1^\circ \subseteq \overline{\mathcal{M}}_g^\circ.$$

Proposition 3.9. *One has that* $F_0 \subseteq \widetilde{\mathcal{M}}_g$ *and* $F_1 \subseteq \widetilde{\mathcal{M}}_g$*.*

Proof. We only show that $F_1 \subseteq \widetilde{\Delta}_1 \subseteq \widetilde{\mathcal{M}}_g$, the argument for F_0 being analogous. To that end, choose a point $[X \cup_y E] \in \Delta_1^\circ$, where X is a general curve of genus $g-1$. Assuming $[X \cup_y E] \in \overline{\mathcal{M}}^r_{g,d-1}$, it follows that $\overline{G}^r_{d-1}(X \cup_y E) \neq \emptyset$. Denoting by $L_X \in \mathrm{Pic}^{d-1}(X)$ the underlying line bundle of the X-aspect of ℓ, we obtain $h^0(X, L_X(-2y)) \geq r$, that is, $W^{r-1}_{d-3}(X) \neq \emptyset$. In both cases (9) and (10), we have $\rho(g-1, r-1, d-3) < 0$, which contradicts the generality of X. The same consideration shows that F_1 is disjoint from both $\mathcal{T}_0$ and $\mathcal{T}_1$. □

We now turn our attention to the pull back $\sigma^*(F_0) \subseteq \widetilde{\mathfrak{G}}^r_d$. We consider the determinantal variety

$$Y := \Big\{(y, L) \in X \times W^r_d(X) : h^0(X, L(-y-q)) \geq r\Big\}, \tag{15}$$

together with the projection $\pi_1 \colon Y \to X$.

Proposition 3.10. *The variety Y is pure of dimension $\rho(g, r, d) + 1$. That is, 3-dimensional in case (9) and 2-dimensional in case (10).*

Proof. We consider the projection $\pi_1 \colon Y \to X$. Its fiber over the point $q \in X$ is the variety of linear series $L \in W^r_d(X)$ having a cusp at q, that is, $h^0(X, L(-2q)) \geq r$. By [EH87, Theorem 1.1], it follows that $\pi_1^{-1}(q)$ has the same dimension as the variety $W^r_d(X_{\mathrm{gen}})$ for a general curve X_{gen} of genus g, which is $\rho(g, r, d)$. Furthermore, using a standard degeneration to a flag curve, it follows that for every point $y \in X$ we have dim $\pi_1^{-1}(y) \leq \rho(g, r, d) + 1$. Therefore each component of Y has dimension $\rho(g, r, d) + 1$. □

Inside Y we introduce the following subvarieties of Y:

$$\Gamma_1 := \Big\{(y, A(y)) : y \in X,\ A \in W^r_{d-1}(X)\Big\} \text{ and}$$

$$\Gamma_2 := \Big\{(y, A(q)) : y \in X,\ A \in W^r_{d-1}(X)\Big\}.$$

These are divisors intersecting transversally along the smooth locus

$$\Gamma := \Big\{(q, A(q)) : A \in W^r_{d-1}(X)\Big\} \cong W^r_{d-1}(X).$$

We then consider the variety obtained from Y by identifying for each $(y, A) \in X \times W^r_{d-1}(X)$, the points $(y, A(y)) \in \Gamma_1$ and $(y, A(q)) \in \Gamma_2$, that is,

$$\widetilde{Y} := Y/[\Gamma_1 \cong \Gamma_2],$$

and denote by $\vartheta : Y \to \widetilde{Y}$ the projection map.

Proposition 3.11. *With notation as above, there is a birational morphism*

$$f : \sigma^*(F_0) \to \widetilde{Y},$$

which is an isomorphism outside $\vartheta(\pi_1^{-1}(q))$. The restriction of f to $f^{-1}\big(\vartheta(\pi_1^{-1}(q))\big)$ forgets the aspect of each limit linear series on the elliptic curve E_∞. Furthermore, both $\mathcal{E}_{|\sigma^*(F_0)}$ and $\mathcal{F}_{|\sigma^*(F_0)}$ are pull backs under f of vector bundles on $\widetilde{Y}$.

Proof. Let $y \in X \setminus \{q\}$ and, as usual, let $\nu \colon X \to X_{yq}$ be the normalization. Recall that we have identified $\sigma^{-1}([X_{yq}])$ with the variety $\overline{W}^r_d(X_{yq}) \subseteq \overline{\mathrm{Pic}}^d(X_{yq})$ of rank 1 torsion-free sheaves on X_{yq} with $h^0(X_{yq}, L) \geq r + 1$. A locally free sheaf $L \in \overline{W}^r_d(X_{yq})$ is uniquely determined by its pull back $\nu^*(L) \in W^r_d(X)$, which has the property that $h^0(X, \nu^*L(-y-q)) = r$. Since X is assumed to be Brill-Noether general $W^r_{d-2}(X) = \emptyset$, so there exists a section of L that does not vanish simultaneously at both y and q. In other words, the 1-dimensional quotient Q of $L_y \oplus L_q$ is uniquely determined as $\nu_*(\nu^*L)/L$.

Assume $L \in \overline{W}^r_d(X_{yq})$ is not locally free, thus $L = \nu_*(A)$ for some line bundle $A \in W^r_{d-1}(X)$. By Remark 3.3, this point corresponds to two points in Y, namely $(y, A(y))$ and $(y, A(q))$. There is a birational morphism $\pi_1^{-1}(y) \to \overline{W}^r_d(X_{yq})$ which is an isomorphism over the locus $W^r_d(X_{yq})$ of locally free sheaves. More precisely, $\overline{W}^r_d(X_{yq})$ is obtained from $\pi_1^{-1}(y)$ by identifying the disjoint divisors $\Gamma_1 \cap \pi_1^{-1}(y)$ and $\Gamma_2 \cap \pi_1^{-1}(y)$.

Finally, when $y = q$, then X_{yq} degenerates to $X \cup_q E_\infty$, where E_∞ is a rational nodal curve. The fiber $\sigma^{-1}\big([X \cup_q E_\infty]\big)$ is the variety of generalized limit linear series $\mathfrak{g}^r_d$ on $X \cup_q E_\infty$ and there is a map $\sigma^{-1}\big([X \cup_q E_\infty]\big) \to \pi^{-1}(q)$ obtained by forgetting the E_∞-aspect of each limit linear series. The statement about the restrictions $\mathcal{E}_{|\sigma^*(F_0)}$ and $\mathcal{F}_{|\sigma^*(F_0)}$ follows from Corollary 3.8 because both restrictions are defined by dropping the information coming from the elliptic tail. □

We now describe the pull back $\sigma^*(F_1) \subseteq \widetilde{\mathfrak{G}}^r_d$. To that end, we define the locus

$$Z := \Big\{(y, L) \in X \times W^r_d(X) : h^0(X, L(-2y)) \geq r\Big\}. \tag{16}$$

By slight abuse of notation, we denote again by $\pi_1 : Z \to X$ the first projection. Arguing along the lines of Proposition 3.10, it follows that Z is pure of dimension $\rho(g, r, d) + 1$.

Proposition 3.12. *The variety Z is an irreducible component of $\sigma^*(F_1)$. Furthermore, we have*

$$c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|\sigma^*(F_1)} = c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|Z} \quad \textit{in case (9), and}$$
$$c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|\sigma^*(F_1)} = c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|Z} \quad \textit{in case (10).}$$

Proof. We deal with the case $(g, r, d) = (23, 6, 26)$, the case (10) being analogous. By the additivity of the Brill-Noether number, if $(\ell_X, \ell_E) \in \sigma^{-1}([X \cup_y E])$ is a limit linear series of type $\mathfrak{g}^6_{26}$, we have that $2 = \rho(23, 6, 26) \geq \rho(\ell_X, y) + \rho(\ell_E, y)$. Since $\rho(\ell_E, y) \geq 0$, we obtain that $\rho(\ell_X, y) \leq 2$. If $\rho(\ell_E, y) = 0$, then $\ell_E = 19y + |\mathcal{O}_E(7y)|$. This shows that ℓ_E is uniquely determined, while the aspect $\ell_X \in G^6_{26}(X)$ is a complete linear series with a cusp at $y \in X$. This gives rise to an element from Z and shows that $Z \times \{\ell_E\} \cong Z$ is a component of $\sigma^*(F_1)$.

The other components of $\sigma^*(F_1)$ are indexed by Schubert indices

$$\alpha := \big(0 \leq \alpha_0 \leq \ldots \leq \alpha_6 \leq 20 = 26 - 6\big),$$

such that lexicographically $\alpha > (0, 1, 1, 1, 1, 1, 1)$, and $7 \leq \sum_{j=0}^{6} \alpha_j \leq 9$, for we must also have $-1 \leq \rho(\ell_X, y) \leq 1$ for any point $y \in X$, see [Far13, Theorem 0.1]. For such an index α, we set $\alpha^c := (20 - \alpha_6, \ldots, 20 - \alpha_0)$ to be the complementary Schubert index, and define

$$Z_\alpha := \big\{(y, \ell_X) \in X \times G^6_{26}(X) : \alpha^{\ell_X}(y) \geq \alpha\big\} \text{ and } W_\alpha := \big\{\ell_E \in G^6_{26}(E) : \alpha^{\ell_E}(y) \geq \alpha^c\big\}.$$

Then the following relation holds

$$\sigma^*(F_1) = Z + \sum_{\alpha > (0,1,1,1,1,1,1)} m_\alpha \, Z_\alpha \times W_\alpha,$$

where the multiplicities m_α can be determined via Schubert calculus but play no role in our calculation. Our claim now follows for dimension reasons. Applying the Brill-Noether Theorem [EH87, Theorem 1.1] in the pointed setting and using that X is a general curve, we obtain the estimate $\dim Z_\alpha = 1 + \rho(22, 6, 26) - \sum_{j=0}^{6} \alpha_j < 3$, for every index $\alpha > (0, 1, 1, 1, 1, 1, 1)$. In the definition of the test curve F_1, the point of attachment $y \in E$ is fixed, therefore the restrictions of both $\mathcal{E}$ and $\mathcal{F}$ are pulled-back from Z_α and one obtains that $c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|Z_\alpha \times W_\alpha} = 0$ for dimension reasons. □

4. The class of the virtual divisor on $\widetilde{\mathcal{M}}_{23}$

In this section we compute the class of $[\widetilde{\mathcal{M}}_{23}]^{\mathrm{virt}}$ and prove the $g = 23$ part of Theorem 1.2.

4.1. **Chern numbers of tautological classes on Jacobians.** We repeatedly use facts about intersection theory on Jacobians, and refer to [ACGH85, Chapters VII–VIII] for background on this topic and to [HM82, Har84, Far09] for applications to divisor class calculations on $\overline{\mathcal{M}}_g$. Let X be a Brill-Noether general curve of genus g. Denote by $\mathcal{P}$ a Poincaré line bundle on $X \times \mathrm{Pic}^d(X)$ and by

$$\pi_1 \colon X \times \mathrm{Pic}^d(X) \to X \text{ and } \pi_2 \colon X \times \mathrm{Pic}^d(X) \to \mathrm{Pic}^d(X)$$

the two projections. We introduce the class $\eta = \pi_1^*([x_0]) \in H^2(X \times \mathrm{Pic}^d(X), \mathbb{Z})$, where $x_0 \in X$ is an arbitrary point. After picking a symplectic basis $\delta_1, \ldots, \delta_{2g} \in H^1(X, \mathbb{Z}) \cong H^1(\mathrm{Pic}^d(X), \mathbb{Z})$, we consider the class

$$\gamma := -\sum_{\alpha=1}^{g} \Big(\pi_1^*(\delta_\alpha)\pi_2^*(\delta_{g+\alpha}) - \pi_1^*(\delta_{g+\alpha})\pi_2^*(\delta_\alpha)\Big) \in H^2(X \times \mathrm{Pic}^d(X), \mathbb{Z}).$$

One has the formula $c_1(\mathcal{P}) = d\cdot\eta+\gamma$, which describes the Künneth decomposition of $c_1(\mathcal{P})$, as well as the relations $\gamma^3=0$, $\gamma\eta=0$, $\eta^2=0$, and $\gamma^2=-2\eta\pi_2^*(\theta)$, see [ACGH85, page 335]. Assuming $W_d^{r+1}(X)=\emptyset$, that is, when the Brill-Noether number $\rho(g,r+1,d)$ is negative (which happens in both cases (9) and (10)), the smooth variety $W_d^r(X)$ admits a rank $r+1$ vector bundle

$$\mathcal{M} := (\pi_2)_*\Big(\mathcal{P}_{|X\times W_d^r(X)}\Big)$$

with fibers $\mathcal{M}(L)\cong H^0(X,L)$, for $L\in W_d^r(X)$. In order to compute the Chern numbers of $\mathcal{M}$, we repeatedly employ the Harris-Tu formula [HT84], which we now explain. We write

$$\sum_{i=0}^{r} c_i(\mathcal{M}^\vee) = (1+x_1)\cdots(1+x_{r+1}).$$

Then, for every class $\zeta\in H^*\big(\mathrm{Pic}^d(X),\mathbb{Z}\big)$, any Chern number $c_{j_1}(\mathcal{M})\cdots c_{j_s}(\mathcal{M})\cdot\zeta\in H^{\mathrm{top}}(W_d^r(X),\mathbb{Z})$ can be computed by using repeatedly the formal identities[1]:

$$x_1^{i_1}\cdots x_{r+1}^{i_{r+1}}\cdot\zeta = \det\Big(\frac{\theta^{g+r-d+i_j-j+k}}{(g+r-d+i_j-j+k)!}\Big)_{1\le j,k\le r+1}\ \zeta. \tag{17}$$

Via the expression of the Vandermonde determinant, (17) leads to the following formula in $H^{\mathrm{top}}(W_d^r(X),\mathbb{Z})$:

$$x_1^{i_1}\cdots x_{r+1}^{i_{r+1}}\cdot\theta^{\rho(g,r,d)-i_1-\cdots-i_{r+1}} = g!\ \frac{\prod_{j>k}(i_k-i_j+j-k)}{\prod_{k=1}^{r+1}(g-d+2r+i_k-k)!}. \tag{18}$$

Jet bundles are employed several times in this section, and we recall their definition. Denote by

$$\mu,\nu\colon X\times X\times \mathrm{Pic}^d(X)\to X\times\mathrm{Pic}^d(X)$$

the two projections and by $\Delta\subseteq X\times X\times\mathrm{Pic}^d(X)$ the diagonal. Then the *jet bundle* of the Poincaré line bundle $\mathcal{P}$ on $X\times\mathrm{Pic}^d(X)$ is defined as $J_1(\mathcal{P}) := \nu_*\big(\mu^*(\mathcal{P})\otimes\mathcal{O}_{2\Delta}\big)$. Its fiber over a point $(y,L)\in X\times\mathrm{Pic}^d(X)$ is naturally identified with $H^0(L\otimes\mathcal{O}_{2y})$.

4.2. **Top intersection products in the Jacobian of a curve of genus 22.** We now specialize to the case of a general curve X of genus 22. By Riemann-Roch the duality $W^6_{26}(X)\cong W^1_{16}(X)$ holds. Since $\rho(22,7,26)=-2<0$, note that $W^7_{26}(X)=\emptyset$, so we can consider the rank 7 tautological vector bundle $\mathcal{M}$ on $W^6_{26}(X)$ with fibers $\mathcal{M}_L\cong H^0(X,L)$. The vector bundle $\mathcal{N} := (R^1\pi_2)_*\Big(\mathcal{P}_{|X\times W^6_{26}(X)}\Big)$ has rank 2 and we explain how its two Chern classes determine all of the Chern classes of $\mathcal{M}$.

Proposition 4.1. *For a general curve X of genus 22 we set $c_i := c_i(\mathcal{M}^\vee)$, for $i=1,\dots,7$, and $y_i := c_i(\mathcal{N})$, for $i=1,2$. Then the following relations hold in $H^*(W^6_{26}(X),\mathbb{Z})$:*

$$c_1 = \theta - y_1 \ \textit{and}$$
$$c_{i+2} = \frac{1}{i!}y_2\theta^i - \frac{1}{(i+1)!}y_1\theta^{i+1} + \frac{1}{(i+2)!}\theta^{i+2}\ \textit{for all } i\ge 0.$$

Proof. Fix an effective divisor $D\in X_e$ of sufficiently large degree e. There is an exact sequence

$$0\to\mathcal{M}\to(\pi_2)_*\Big(\mathcal{P}\otimes\mathcal{O}(\pi^*D)\Big)\to(\pi_2)_*\Big(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\Big)\to R^1\pi_{2*}\Big(\mathcal{P}_{|X\times W^6_{26}(X)}\Big)\to 0.$$

[1]Formula (17) is to be interpreted as a *formal recipe* for evaluating the Chern numbers $c_{j_1}(\mathcal{M})\cdots c_{j_s}(\mathcal{M})\ \cdot\zeta$. Precisely, $W_d^r(X)$ can be expressed as the degeneracy locus of a morphism of vector bundles $\mathcal{V}_1\to\mathcal{V}_2$ over $\mathrm{Pic}^d(X)$ and $\mathcal{M}$ is the kernel bundle of the restriction of this map to $W_d^r(X)$. Passing to a flag variety $\alpha\colon\mathbb{F}:=F(\mathcal{V}_1)\to\mathrm{Pic}^d(X)$ over which one has canonical choices for the Chern roots $x_1,\dots,x_{r+1}$, formula (17) is then proven in [HT84, Corollary 2.6] at the level of $\mathbb{F}$.

Recall that $\mathcal{N}$ is the vector bundle on the right in the exact sequence above. By [ACGH85, Chapter VII], we have $c_{\rm tot}\Big((\pi_2)_*(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D))\Big) = e^{-\theta}$, and the total Chern class of the vector bundle $(\pi_2)_*\Big(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\Big)$ is trivial. We therefore obtain

$$c_{\rm tot}(\mathcal{N})\cdot e^{-\theta} = \sum_{i=0}^{8}(-1)^i c_i.$$

Hence $c_{i+2} = \frac{1}{i!}y_2\theta^i - \frac{1}{(i+1)!}y_i\theta^{i+1} + \frac{1}{(i+2)!}\theta^{i+2}$ for all $i\geq 0$, as desired. □

Using Proposition 4.1, any Chern number on the smooth 8-fold $W^6_{26}(X)$ can be expressed in terms of monomials in the classes u_1, u_2, and θ, where u_1 and u_2 are the Chern roots of $\mathcal{N}$, that is,

$$y_1 = c_1(\mathcal{N}) = u_1+u_2 \ \text{ and } y_2 = c_2(\mathcal{N}) = u_1\cdot u_2.$$

We record for further use the following formal identities on $H^{\rm top}(W^6_{26}(X),\mathbb{Z})$, which are obtained by applying formula (17) in the case $g=22$, $r=1$ and $d=16$, using the canonical isomorphism $H^1(X,L)\cong H^0\big(X,\omega_X\otimes L^\vee\big)^\vee$ provided by Serre duality.

$$u_1^3\theta^5 = \frac{4\cdot 22!}{11!\cdot 7!},\quad u_2^3\theta^5 = -\frac{2\cdot 22!}{8!\cdot 10!},\quad u_1^2\theta^6 = \frac{3\cdot 22!}{10!\cdot 7!},\quad u_2^2\theta^6 = -\frac{22!}{8!\cdot 9!},\quad u_1\theta^7 = \frac{2\cdot 22!}{7!\cdot 9!},$$
$$u_2\theta^7 = 0,\quad u_1u_2^4\theta^3 = -\frac{2\cdot 22!}{9!\cdot 11!},\quad u_1^4u_2\theta^3 = \frac{4\cdot 22!}{8!\cdot 12!},\quad u_1^2u_2\theta^5 = \frac{2\cdot 22!}{8!\cdot 10!},\quad u_1u_2^2\theta^5 = 0,$$
$$u_1^2u_2^3\theta^3 = 0,\quad u_1^3u_2^2\theta^3 = \frac{2\cdot 22!}{9!\cdot 11!},\quad u_1^2u_2^2\theta^4 = \frac{22!}{9!\cdot 10!},\quad u_1^4\theta^4 = \frac{5\cdot 22!}{7!\cdot 12!},\quad u_2^4\theta^4 = -\frac{3\cdot 22!}{8!\cdot 11!},$$
$$u_1^3u_2\theta^4 = \frac{3\cdot 22!}{8!\cdot 11!},\quad u_1u_2^3\theta^4 = -\frac{22!}{9!\cdot 10!},\quad u_1u_2\theta^6 = \frac{22!}{8!\cdot 9!},\quad \theta^8 = \frac{22!}{7!\cdot 8!}.$$

To compute the corresponding Chern numbers on $W^6_{26}(X)$, one uses Proposition 4.1 and the previous formulas. Each Chern number corresponds to a degree 8 polynomial in u_1, u_2, and θ, which is symmetric in u_1 and u_2.

We now compute the classes of the loci Y and Z appearing in Propositions 3.11 and 3.12.

Proposition 4.2. *Let $[X,q]$ be a general 1-pointed curve of genus 22, let $\mathcal{M}$ denote the tautological rank 7 vector bundle over $W^6_{26}(X)$, and let $c_i = c_i(\mathcal{M}^\vee)\in H^{2i}(W^6_{26}(X),\mathbb{Z})$ as before. Then the following hold:*

1. $[Z] = \pi_2^*(c_6) - 6\eta\theta\pi_2^*(c_4) + (94\eta+2\gamma)\pi_2^*(c_5)\in H^{12}(X\times W^6_{26}(X),\mathbb{Z})$.
2. $[Y] = \pi_2^*(c_6) - 2\eta\theta\pi_2^*(c_4) + (25\eta+\gamma)\pi_2^*(c_3)\in H^{12}(X\times W^6_{26}(X),\mathbb{Z})$.

Proof. Recall that $W^6_{26}(X)$ is smooth of dimension 8. We realize the locus Z defined by (16) as the degeneracy locus of a vector bundle morphism over $X\times W^6_{26}(X)$. Precisely, for each pair $(y,L)\in X\times W^6_{26}(X)$, there is a natural map

$$H^0(X,L\otimes\mathcal{O}_{2y})^\vee\to H^0(X,L)^\vee,$$

which globalizes to a bundle morphism $\zeta\colon J_1(\mathcal{P})^\vee\to\pi_2^*(\mathcal{M})^\vee$ over $X\times W^6_{26}(X)$. Then we have the identification $Z = Z_1(\zeta)$, that is, Z is the first degeneracy locus of ζ. The Porteous formula yields $[Z] = c_6\Big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\Big)$. To evaluate this class, we use the exact sequence over $X\times\mathrm{Pic}^{26}(X)$ involving the jet bundle:

$$0\longrightarrow\pi_1^*(\omega_X)\otimes\mathcal{P}\longrightarrow J_1(\mathcal{P})\longrightarrow\mathcal{P}\longrightarrow 0.$$

We compute the total Chern class of the formal inverse of the jet bundle as follows:

$$\begin{aligned}c_{\rm tot}(J_1(\mathcal{P})^\vee)^{-1} &= \Big(\sum_{j\geq 0}(\deg(L)\eta+\gamma)^j\Big)\cdot\Big(\sum_{j\geq 0}\big((2g(X)-2+\deg(L))\eta+\gamma\big)^j\Big)\\ &= \big(1+26\eta+\gamma+\gamma^2+\cdots\big)\cdot\big(1+68\eta+\gamma+\gamma^2+\cdots\big) = 1+94\eta+2\gamma-6\eta\theta,\end{aligned}$$

leading to the desired formula for $[Z]$.

To compute the class of the variety Y defined in (15) we proceed in a similar way. Recall that

$$\mu, \nu \colon X \times X \times \mathrm{Pic}^{26}(X) \to X \times \mathrm{Pic}^{26}(X)$$

denote the two projections and $\Delta \subseteq X \times X \times \mathrm{Pic}^{26}(X)$ is the diagonal. Set $\Gamma_q := \{q\} \times \mathrm{Pic}^{26}(X)$. We introduce the rank 2 vector bundle $\mathcal{B} := \mu_*\big(\nu^*(\mathcal{P}) \otimes \mathcal{O}_{\Delta+\nu^*(\Gamma_q)}\big)$ over $X \times W^6_{26}(X)$. Note that there is a bundle morphism $\chi : \mathcal{B}^\vee \to (\pi_2)^*(\mathcal{M})^\vee$ such that $Y = Z_1(\chi)$. Since we also have that

$$c_{\mathrm{tot}}(\mathcal{B}^\vee)^{-1} = \Big(1 + (\deg(L)\eta + \gamma) + (\deg(L)\eta + \gamma)^2 + \cdots\Big) \cdot \big(1 - \eta\big) = 1 + 25\eta + \gamma - 2\eta\theta,$$

we immediately obtain the stated expression for $[Y]$. □

The following formulas are applications of Grothendieck-Riemann-Roch.

Proposition 4.3. *Let X be a general curve of genus 22, let $q \in X$ be a fixed point, and consider the vector bundles $\mathcal{A}_2$ and $\mathcal{B}_2$ on $X \times \mathrm{Pic}^d(X)$ having fibers*

$$\mathcal{A}_2(y, L) = H^0\big(X, L^{\otimes 2}(-2y)\big) \quad \textit{and} \quad \mathcal{B}_2(y, L) = H^0\big(X, L^{\otimes 2}(-y - q)\big),$$

respectively. One then has the following formulas:

$$c_1(\mathcal{A}_2) = -4\theta - 4\gamma - 146\eta, \qquad c_1(\mathcal{B}_2) = -4\theta - 2\gamma - 51\eta,$$
$$c_2(\mathcal{A}_2) = 8\theta^2 + 560\eta\theta + 16\gamma\theta, \qquad c_2(\mathcal{B}_2) = 8\theta^2 + 196\eta\theta + 8\theta\gamma,$$
$$c_3(\mathcal{A}_2) = -\frac{32}{3}\theta^3 - 1072\eta\theta^2 - 32\theta^2\gamma, \qquad c_3(\mathcal{B}_2) = -\frac{32}{3}\theta^3 - 376\eta\theta^2 - 16\theta^2\gamma.$$

Proof. This is an immediate application of Grothendieck-Riemann-Roch with respect to the projection map $\nu \colon X \times X \times \mathrm{Pic}^{26}(X) \to X \times \mathrm{Pic}^{26}(X)$. Since $H^1(X, L^{\otimes 2}(-2y)) = 0$ for every $(y, L) \in X \times \mathrm{Pic}^{26}(X)$, the vector bundle $\mathcal{A}_2$ is realized as a push forward under the map ν:

$$\mathcal{A}_2 = \nu_!\Big(\mu^*\big(\mathcal{P}^{\otimes 2} \otimes \mathcal{O}_{X\times X\times \mathrm{Pic}^{26}(X)}(-2\Delta)\big)\Big) = \nu_*\Big(\mu^*\big(\mathcal{P}^{\otimes 2} \otimes \mathcal{O}_{X\times X\times \mathrm{Pic}^{26}(X)}(-2\Delta)\big)\Big),$$

and we apply Grothendieck-Riemann-Roch to ν. One finds $\mathrm{ch}_2(\mathcal{A}_2) = 8\eta\theta$ and $\mathrm{ch}_n(\mathcal{A}_2) = 0$ for $n \geq 3$. Furthermore, $\nu_*(c_1(\mathcal{P})^2) = -2\theta$. One then obtains $c_1(\mathcal{A}_2) = -4\theta - 4\gamma - (4d + 2g - 4)\eta$, which then yields the formula for $c_2(\mathcal{A}_2)$. Since $\mathrm{ch}_3(\mathcal{A}_2) = 0$, we find that $c_3(\mathcal{A}_2) = c_1(\mathcal{A}_2)c_2(\mathcal{A}_2) - \frac{c_1^3(\mathcal{A}_2)}{3}$, which by substitution leads to the claimed expression.

The calculation of $\mathcal{B}_2$ is similar. We find that $c_1(\mathcal{B}_2) = -4\theta - 2\gamma - (2d - 1)\eta$ and $\mathrm{ch}_2(\mathcal{B}_2) = 4\eta\theta$, whereas $\mathrm{ch}_n(\mathcal{B}_2) = 0$ for $n \geq 3$. □

4.3. **The slope computation.** In this section we complete the calculation of the virtual class $[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}}$. We shall use repeatedly that if $\mathcal{V}$ is a vector bundle of rank $r+1$ on a stack, the Chern classes of its second symmetric product can be computed as follows:

(1) $c_1(\mathrm{Sym}^2(\mathcal{V})) = (r + 2)c_1(\mathcal{V})$,
(2) $c_2(\mathrm{Sym}^2(\mathcal{V})) = \frac{r(r+3)}{2}c_1^2(\mathcal{V}) + (r + 3)c_2(\mathcal{V})$,
(3) $c_3(\mathrm{Sym}^2(\mathcal{V})) = \frac{r(r+4)(r-1)}{6}c_1^3(\mathcal{V}) + (r + 5)c_3(\mathcal{V}) + (r^2 + 4r - 1)c_1(\mathcal{V})c_2(\mathcal{V})$.

We expand the virtual class

$$[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}} = \sigma_*\Big(c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\Big) = a\lambda - b_0\delta_0 - b_1\delta_1 \in CH^1(\widetilde{\mathcal{M}}_{23}).$$

Our task is to determine the coefficients a, b_0 and b_1. We begin with the coefficient of δ_1.

Theorem 4.4. *Let X be a general curve of genus 22 and denote by $F_1 \subseteq \widetilde{\Delta}_1 \subseteq \widetilde{\mathcal{M}}_{23}$ the associated test curve. Then the coefficient of δ_1 in the expansion of $[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}}$ is equal to*

$$b_1 = \frac{1}{2g(X) - 2}\sigma^*(F_1) \cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) = 13502337992 = \frac{4}{9}\binom{19}{8}401951.$$

Proof. We intersect the degeneracy locus of the map $\phi\colon \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with the 3-fold $\sigma^*(F_1)$. By Proposition 3.12, we have

$$\sigma^*(F_1)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big) = c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)_{|Z} = c_3(\mathcal{F}_{|Z}) - c_3(\mathrm{Sym}^2\mathcal{E}_{|Z}) - c_1(\mathcal{F}_{|Z})c_2(\mathrm{Sym}^2\mathcal{E}_{|Z})$$

$$+2c_1(\mathrm{Sym}^2\mathcal{E}_{|Z})c_2(\mathrm{Sym}^2\mathcal{E}_{|Z}) - c_1(\mathrm{Sym}^2\mathcal{E}_{|Z})c_2(\mathcal{F}_{|Z}) + c_1^2(\mathrm{Sym}^2\mathcal{E}_{|Z})c_1(\mathcal{F}_{|Z}) - c_1^3(\mathrm{Sym}^2\mathcal{E}_{|Z}).$$

We now evaluate the terms that appear in the righthand side of this expression.

In the course of proving Proposition 4.2, we constructed a morphism $\zeta\colon J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$ of vector bundles on Y globalizing the maps $H^0(\mathcal{O}_{2y})^\vee \to H^0(X,L)^\vee$. The kernel sheaf $\mathrm{Ker}(\zeta)$ is locally free of rank 1. If U is the line bundle on Z with fiber

$$U(y,L) = \frac{H^0(X,L)}{H^0(X,L(-2y))} \hookrightarrow H^0(X, L\otimes \mathcal{O}_{2y})$$

over a point $(y,L)\in Z$, then one has the following exact sequence over Z:

$$0 \longrightarrow U \longrightarrow J_1(\mathcal{P}) \longrightarrow \big(\mathrm{Ker}(\zeta)\big)^\vee \longrightarrow 0.$$

In particular, by Proposition 4.2, we find that

$$c_1(U) = 2\gamma + 94\eta + c_1(\mathrm{Ker}(\zeta)). \tag{19}$$

The products of the Chern class of $\mathrm{Ker}(\zeta)$ with other classes coming from $X\times W^6_{26}(X)$ can be computed from the formula in [HT84]:

$$c_1\big(\mathrm{Ker}(\zeta)\big)\cdot \xi_{|Z} = -c_7\Big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\Big)\cdot \xi_{|Z} \tag{20}$$

$$= -\Big(\pi_2^*(c_7) - 6\eta\theta\pi_2^*(c_5) + (94\eta+2\gamma)\pi_2^*(c_6)\Big)\cdot \xi_{|Z},$$

where $\xi\in H^2(X\times W^6_{26}(X),\mathbb{Z})$.

If $\mathcal{A}_3$ denotes the rank 31 vector bundle on Z having fibers

$$\mathcal{A}_3(y,L) = H^0(X, L^{\otimes 2})$$

constructed as a push forward of a line bundle on $X\times X\times \mathrm{Pic}^{26}(X)$, then $U^{\otimes 2}$ can be embedded in $\mathcal{A}_3/\mathcal{A}_2$. We consider the quotient

$$\mathcal{G} := \frac{\mathcal{A}_3/\mathcal{A}_2}{U^{\otimes 2}}.$$

The morphism $U^{\otimes 2}\to \mathcal{A}_3/\mathcal{A}_2$ vanishes along the locus of pairs (y,L) where L has a base point. This implies that $\mathcal{G}$ has torsion along the locus $\Gamma\subseteq Z$ consisting of pairs $(q, A(q))$, where $A\in W^6_{25}(X)$. Furthermore, $\mathcal{F}_{|Z}$ is identified as a subsheaf of $\mathcal{A}_3$ with the kernel of the map $\mathcal{A}_3\to\mathcal{G}$. Summarizing, there is an exact sequence of vector bundles on Z

$$0 \longrightarrow \mathcal{A}_{2|Z} \longrightarrow \mathcal{F}_{|Z} \longrightarrow U^{\otimes 2} \longrightarrow 0. \tag{21}$$

Over a general point $(y,L)\in Z$, this sequence reflects the decomposition

$$\mathcal{F}(y,L) = H^0(X, L^{\otimes 2}(-2y)) \oplus K\cdot u^2,$$

where $u\in H^0(X,L)$ is a section such that $\mathrm{ord}_y(u)=1$.

Hence using the exact sequence (21), one computes:

$c_1(\mathcal{F}_{|Z}) = c_1(\mathcal{A}_{2|Z}) + 2c_1(U)$, $\quad c_2(\mathcal{F}_{|Z}) = c_2(\mathcal{A}_{2|Z}) + 2c_1(\mathcal{A}_{2|Z})c_1(U)$ and
$c_3(\mathcal{F}_{|Z}) = c_3(\mathcal{A}_2) + 2c_2(\mathcal{A}_{2|Z})c_1(U)$.

Recalling that $\mathcal{E}_{|Z} = \pi_2^*(\mathcal{M})_{|Z}$, we obtain that:

$$\begin{aligned}\sigma^*(F_1)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2\mathcal{E}\big) &= c_3(\mathcal{A}_{2|Z}) + c_2(\mathcal{A}_{2|Z})c_1(U^{\otimes 2}) - c_3(\mathrm{Sym}^2\pi_2^*\mathcal{M}_{|Z})\\ &\quad -\Big(\frac{r(r+3)}{2}c_1(\pi_2^*\mathcal{M}_{|Z}) + (r+3)c_2(\pi_2^*\mathcal{M}_{|Z})\Big)\cdot\Big(c_1(\mathcal{A}_{2|Z}) + c_1(U^{\otimes 2}) - 2(r+2)c_1(\pi_2^*\mathcal{M}_{|Z})\Big)\\ &\quad -(r+2)c_1(\pi_2^*\mathcal{M}_{|Z})c_2(\mathcal{A}_{2|Z}) - (r+2)c_1(\pi_2^*\mathcal{M}_{|Z})c_1(\mathcal{A}_{2|Z})c_1(U^{\otimes 2})\\ &\quad +(r+2)^2c_1^2(\pi_2^*\mathcal{M}_{|Z})c_1(\mathcal{A}_{2|Z}) + (r+2)^2c_1^2(\pi_2^*\mathcal{M}_{|Z})c_1(U^{\otimes 2}) - (r+2)^3c_1^3(\pi_2^*\mathcal{M}_{|Z}).\end{aligned}$$

Here, $c_i(\pi_2^*\mathcal{M}_{|Z}^{\vee}) = \pi_2^*(c_i) \in H^{2i}(Z,\mathbb{Z})$ and $r = \mathrm{rk}(\mathcal{M}) - 1 = 6$. The Chern classes of $\mathcal{A}_{2|Z}$ are obtained by applying Proposition 4.3. Recall that in (19) we expressed $c_1(U)$ in terms of $c_1((\mathrm{Ker}(\zeta))$ and the classes η and γ. Substituting (19) for $c_1(U)$, when expanding $\sigma^*(F_1)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)$, one distinguishes between terms that do and those that do not contain the first Chern class of $\mathrm{Ker}(\zeta)$. The coefficient of $c_1\big(\mathrm{Ker}(\zeta)\big)$ in $\sigma^*(F_1)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)$ is evaluated using (20). First we consider the part of this product that *does not* contain $c_1\big(\mathrm{Ker}(\zeta)\big)$, and we obtain

$$\begin{aligned}&36\pi_2^*(c_2)\theta - 148\pi_2^*(c_1^2)\theta + 1554\eta\pi_2^*(c_1^2) - 85\pi_2^*(c_1c_2) - \frac{32}{3}\theta^3 + 304\eta\theta^2 - 1280\eta\theta\pi_2^*(c_1)\\ &\qquad +130\pi_2^*(c_1^3) - 378\eta\pi_2^*(c_2) + 64\theta^2\pi_2^*(c_1) + 11\pi_2^*(c_3) \in H^6\big(X\times W^6_{26}(X),\mathbb{Z}\big).\end{aligned}$$

This polynomial of degree 3 gets multiplied by the class $[Z]$, expressed as the degree 6 polynomial in θ, η, and $\pi_2^*(c_i)$ obtained in Proposition 4.2. Adding to it the contribution coming from $c_1\big(\mathrm{Ker}(\zeta)\big)$, one obtains a homogeneous polynomial of degree 9 in η, θ, and $\pi_2^*(c_i)$ for $i = 1,\ldots,7$. The only nonzero monomials are those containing η. After retaining only these monomials and dividing by η, the resulting degree 8 polynomial in θ, $c_i \in H^*(W^6_{26}(X),\mathbb{Z})$ can be brought to a manageable form using Proposition 4.1. After lengthy but straightforward manipulations carried out using *Maple*, one finds

$$\begin{aligned}\sigma^*(F_1)\cdot c_3\big(\mathrm{Sym}^2(\mathcal{E})-\mathcal{F}\big) = \eta\pi_2^*\Big(&-780c_1^3c_4\theta + 12220c_1^3c_5 + 888c_1^2c_4\theta^2 - 13468c_1^2c_5\theta - 5402c_1^2c_6\\ &- 384\theta^3c_1c_4 + 5632\theta^2c_1c_5 + 510\theta c_1c_2c_4 + 4480c_1c_6\theta - 7990c_1c_2c_5\\ &+ 2336c_1c_7 - 216c_2c_4\theta^2 + 3276c_2c_5\theta - 66c_3c_4\theta + 1034c_3c_5 + 1314c_2c_6\\ &+ 64c_4\theta^4 - \frac{2720}{3}c_5\theta^3 - 1072c_6\theta^2 - 1120c_7\theta\Big).\end{aligned}$$

We suppress η and the remaining polynomial lives inside $H^{16}(W^6_{26}(X),\mathbb{Z})$. Using (17), we explicitly calculate all top Chern numbers on $W^6_{26}(X)$ and we eventually find that

$$b_1 = \frac{1}{42}\sigma^*(F_1)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big) = 13502337992,$$

as required. □

Theorem 4.5. *Let $[X,q]$ be a general pointed curve of genus 22 and let $F_0 \subseteq \widetilde{\Delta}_0 \subseteq \widetilde{\mathcal{M}}_{23}$ be the associated test curve. Then*

$$\sigma^*(F_0)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big) = 44b_0 - b_1 = 93988702808.$$

It follows that $b_0 = \frac{4}{9}\binom{19}{8}72725$.

Proof. By Proposition 3.11, the vector bundles $\mathcal{E}_{|\sigma^*(F_0)}$ and $\mathcal{F}_{|\sigma^*(F_0)}$ are both pull backs of vector bundles on $\widetilde{Y} = Y/[\Gamma_1 \sim \Gamma_2]$. By abuse of notation we denote these vector bundles by the same symbols, that is, we have $\mathcal{E}_{|\sigma^*(F^0)} = f^*(\mathcal{E}_{|\widetilde{Y}})$ and $\mathcal{F}_{|\sigma^*(F_0)} = f^*(\mathcal{F}_{|\widetilde{Y}})$. Following broadly the proof of Theorem 4.4, we evaluate the terms appearing in $\sigma^*(F_0)\cdot c_3(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})) = c_3\big(\mathcal{F}_{|Y} - \mathrm{Sym}^2(\mathcal{E}_{|Y}\big)$, where $\mathcal{E}_{|Y} = \vartheta^*(\mathcal{E}_{|\widetilde{Y}})$ and $\mathcal{F}_{|Y} = \vartheta^*(\mathcal{F}_{|\widetilde{Y}})$ respectively.

Let V be the line bundle on Y with fiber

$$V(y,L)=\frac{H^0(X,L)}{H^0(X,L(-y-q))}\hookrightarrow H^0(X,L\otimes\mathcal{O}_{y+q})$$

over a point $(y,L)\in Y$. There is an exact sequence of vector bundles over Y

$$0\longrightarrow V\longrightarrow\mathcal{B}\longrightarrow\big(\mathrm{Ker}(\chi)\big)^{\vee}\longrightarrow 0,$$

where $\chi\colon\mathcal{B}^{\vee}\to\pi_2^*(\mathcal{M})^{\vee}$ is the bundle morphism defined in the second part of the proof of Proposition 4.2. In particular, $c_1(V)=25\eta+\gamma+c_1\big(\mathrm{Ker}(\chi)\big)$, for the Chern class of $\mathcal{B}$ has been computed in the proof of Proposition 4.2. By using again [HT84], we find the following formulas for the Chern numbers of $\mathrm{Ker}(\chi)$:

$$c_1(\mathrm{Ker}(\chi))\cdot\xi_{|Y}=-c_7\big(\pi_2^*(\mathcal{M})^{\vee}-\mathcal{B}^{\vee}\big)\cdot\xi_{|Y}=-\Big(\pi_2^*(c_7)+\pi_2^*(c_6)(13\eta+\gamma)-2\pi_2^*(c_4)\eta\theta\Big)\cdot\xi_{|Y},$$

for any class $\xi\in H^2(X\times W^6_{26}(X),\mathbb{Z})$. We have previously defined the vector bundle $\mathcal{B}_2$ over $C\times W^6_{26}(X)$ with fiber $\mathcal{B}_2(y,L)=H^0(X,L^{\otimes 2}(-y-q))$. We show that there is an exact sequence of bundles over Y

$$0\longrightarrow\mathcal{B}_{2|Y}\longrightarrow\mathcal{F}_{|Y}\longrightarrow V^{\otimes 2}\longrightarrow 0. \tag{22}$$

If $\mathcal{B}_3$ is the vector bundle on Y with fibers $\mathcal{B}_3(y,L)=H^0(X,L^{\otimes 2})$, we have an injective morphism of sheaves $V^{\otimes 2}\hookrightarrow\mathcal{B}_3/\mathcal{B}_2$ locally given by

$$v^{\otimes 2}\mapsto v^2 \bmod H^0(X,L^{\otimes 2}(-y-q)),$$

where $v\in H^0(X,L)$ is any section not vanishing at q and y. Then $\mathcal{F}_{|Y}$ is canonically identified with the kernel of the projection morphism

$$\mathcal{B}_3\to\frac{\mathcal{B}_3/\mathcal{B}_2}{V^{\otimes 2}}$$

and the exact sequence (22) now becomes clear. Therefore

$$c_1(\mathcal{F}_{|Y})=c_1(\mathcal{B}_{2|Y})+2c_1(V),\qquad c_2(\mathcal{F}_{|Y})=c_2(\mathcal{B}_{2|Y})+2c_1(\mathcal{B}_{2|Y})c_1(V)\text{ and}$$
$$c_3(\mathcal{F}_{|Y})=c_3(\mathcal{B}_{2|Y})+2c_2(\mathcal{B}_{2|Y})c_1(V).$$

The part of the intersection number $\sigma^*(F_0)\cdot c_3(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E}))$ *not* containing $c_1\big(\mathrm{Ker}(\chi)\big)$ equals

$$36\pi_2^*(c_2)\theta-148\pi_2^*(c_1^2)\theta-37\eta\pi_2^*(c_1^2)-85\pi_2^*(c_1c_2)-\frac{32}{3}\theta^3-8\eta\theta^2+32\eta\theta\pi_2^*(c_1)$$
$$+130\pi_2^*(c_1^3)+9\eta\pi_2^*(c_2)+64\theta^2\pi_2^*(c_1)+11\pi_2^*(c_3)\in H^6\big(X\times W^6_{26}(X),\mathbb{Z}\big).$$

We multiply this expression by the class $[Y]$ computed in Proposition 4.2. The coefficient of $c_1\big(\mathrm{Ker}(\chi)\big)$ in $\sigma^*(F_0)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)$ equals

$$-2c_2(\mathcal{B}_{2|Y})-2(r+2)^2\pi_2^*(c_1^2)-2(r+2)c_1(\mathcal{B}_{2|Y})\pi_2^*(c_1)+r(r+3)\pi_2^*(c_1^2)+2(r+3)\pi_2^*(c_2),$$

where recall that $r=\mathrm{rk}(\mathcal{M})-1=6$. All in all, we find

$$44b_0-b_1=\sigma^*(F_0)\cdot c_3(\mathcal{F}-\mathrm{Sym}^2\mathcal{E})=\eta\pi_2^*\Big(-260c_1^3c_4\theta+3250c_1^3c_5+296c_1^2c_4\theta^2-3552c_1^2c_5\theta$$
$$-1887c_1^2c_6-128\theta^3c_1c_4+1472\theta^2c_1c_5+170\theta c_1c_2c_4+1568c_1c_6\theta-2125c_1c_2c_5$$
$$+816c_1c_7-72c_2c_4\theta^2+864c_2c_5\theta-22c_3c_4\theta+275c_3c_5+459c_2c_6+\frac{64}{3}c_4\theta^4$$
$$-\frac{704}{3}c_5\theta^3-376c_6\theta^2-392c_7\theta\Big)\in H^{18}\big(X\times W^6_{26}(X),\mathbb{Z}\big).$$

We evaluate each term in this expression by first deleting η and then using (17). □

The following result follows from the definition of the vector bundles $\mathcal{E}$ and $\mathcal{F}$ given in Proposition 3.6. It will provide the third relation between the coefficients of $[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}}$, and thus complete the calculation of its slope.

Theorem 4.6. *Let $[X,q]$ be a general 1-pointed curve of genus 22 and $F_{\mathrm{ell}} \subseteq \widetilde{\mathcal{M}}_{23}$ be the pencil obtained by attaching at the fixed point $q \in X$ a pencil of plane cubics at one of the base points of the pencil. Then one has the relation*

$$a - 12b_0 + b_1 = F_{\mathrm{ell}} \cdot \sigma_* c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) = 0.$$

Proof. Since the genus $g-1$ aspect of each curve in F_{ell} does not vary, it follows from Corollary 3.8 that the vector bundles $\mathcal{E}_{|\sigma^*(F_{\mathrm{ell}})}$ and $\mathcal{F}_{|\sigma^*(F_{\mathrm{ell}})}$ are both trivial, therefore $c_i\big(\mathcal{E}_{|\sigma^*(F_{\mathrm{ell}})}\big) = 0$ and $c_i\big(\mathcal{F}_{|\sigma^*(F_{\mathrm{ell}})}\big) = 0$ for $i \geq 1$, from which the conclusion follows. □

Proof of Theorem 1.2 for $[\widetilde{\mathfrak{D}}_{23}]^{\mathrm{virt}}$. By Theorems 4.4 and 4.5, we have

$$b_0 = \frac{4}{9}\binom{19}{8}72725 \text{ and } b_1 = \frac{4}{9}\binom{19}{8}401951.$$

Combined with Theorem 4.6, we obtain

$$a = \frac{4}{9}\binom{19}{8}470749,$$

and the result follows. □

5. The class of the virtual divisor on $\widetilde{\mathcal{M}}_{2s^2+s+1}$

In this section we prove Theorem 1.7. In particular, we determine the class $[\widetilde{\mathfrak{D}}_{22}]^{\mathrm{virt}}$ that will ultimately be used in the proof that $\overline{\mathcal{M}}_{22}$ is of general type.

5.1. **Top intersection products in the Jacobian of a curve of genus $2s^2+s$.** We next turn our attention to the top intersection products on $W^{2s}_{2s^2+2s+1}(X)$, when X is a general curve of genus $2s^2+s$, for $s \geq 2$. We apply (17) systematically. Our computations are analogous to those in §4, and in many cases we omit the details. Observe that $\rho(2s^2+s, 2s, 2s^2+2s) = 0$, so $W^{2s}_{2s^2+2s}(X)$ is reduced and 0-dimensional. We denote by

$$C_{2s+1} := \frac{(2s^2+s)!\ (2s)!\ (2s-1)!\cdots 2!\ 1!}{(3s)!\ (3s-1)!\cdots(s+1)!\ s!} = \#\Big(W^{2s}_{2s^2+2s}(X)\Big). \tag{23}$$

Moreover $\rho(2s^2+s, 2s+1, 2s^2+2s+1) = -s < 0$, hence it follows that $W^{2s+1}_{2s^2+2s+1}(X) = \emptyset$ and we can consider the tautological rank $2s+1$ vector bundle $\mathcal{M}$ over $W^{2s}_{2s^2+2s+1}(X)$. We write $\sum_{i=0}^{2s+1} c_i(\mathcal{M}^\vee) = (1+x_1)\cdots(1+x_{2s+1})$. We collect the following formulas obtained by applying the Harris-Tu formula (18):

Proposition 5.1. *Let X be as above and set $c_i := c_i(\mathcal{M}^\vee) \in H^{2i}\big(W^{2s}_{2s^2+2s+1}(X), \mathbb{Z}\big)$ to be the Chern classes of the dual of the tautological bundle on $W^{2s}_{2s^2+2s+1}(X)$. The following hold:*

$$\begin{aligned}
c_{2s+1} &= x_1x_2\cdots x_{2s+1} = C_{2s+1},\\
c_{2s}\cdot c_1 &= x_1x_2\cdots x_{2s+1} + x_1^2x_2\cdots x_{2s},\\
c_{2s-1}\cdot c_2 &= x_1x_2\cdots x_{2s+1} + x_1^2x_2\cdots x_{2s} + x_1^2x_2^2x_3\cdots x_{2s-1},\\
c_{2s-1}\cdot c_1^2 &= x_1x_2\cdots x_{2s+1} + 2x_1^2x_2\cdots x_{2s} + x_1^2x_2^2x_3\cdots x_{2s-1} + x_1^3x_2x_3\cdots x_{2s-1},\\
c_{2s}\cdot\theta &= x_1x_2\cdots x_{2s}\cdot\theta = (2s+1)s\ C_{2s+1},\\
c_{2s-1}\cdot c_1\cdot\theta &= x_1x_2\cdots x_{2s}\cdot\theta + x_1^2x_2\cdots x_{2s-1}\cdot\theta,\\
c_{2s-2}\cdot c_2\cdot\theta &= x_1x_2\cdots x_{2s}\cdot\theta + x_1^2x_2\cdots x_{2s-1}\cdot\theta + x_1^2x_2^2x_3\cdots x_{2s-2}\cdot\theta,\\
c_{2s-2}\cdot c_1^2\cdot\theta &= x_1x_2\cdots x_{2s}\cdot\theta + 2x_1^2x_2\ldots x_{2s-1}\cdot\theta + x_1^2x_2^2x_3\cdots x_{2s-2}\cdot\theta + x_1^3x_2x_3\cdots x_{2s-2}\cdot\theta,\\
c_{2s-1}\cdot\theta^2 &= x_1x_2\cdots x_{2s-1}\cdot\theta^2,\\
c_{2s-2}\cdot c_1\cdot\theta^2 &= x_1x_2\cdots x_{2s-1}\cdot\theta^2 + x_1^2x_2\cdots x_{2s-2}\cdot\theta^2.
\end{aligned}$$

Proof. This amounts to a repeated application of (18) and evaluating the corresponding determinants. The right hand side of each formula retains the non-zero terms that appear in the corresponding Chern number. To give an example, we evaluate $c_{2s} \cdot c_1$. Using (18), each monomial $x_1^{i_1} x_2^{i_2} \cdots x_{2s+1}^{i_{2s+1}}$ will vanish as long as there exists a pair $k < j$ such that $i_j - i_k = j - k$. We compute

$$c_{2s} \cdot c_1 = (2s+1)x_1 \cdots x_{2s+1} + x_2^2 x_3 \cdots x_{2s+1} + x_1 x_3^2 x_4 \cdots x_{2s+1} + \cdots + x_1 \cdots x_{2s-1} x_{2s+1}^2 + x_1^2 x_2 \cdots x_{2s}.$$

All the other terms vanish. Then by (18), evaluating each determinant we observe that

$$x_2^2 x_3 \cdots x_{2s+1} = \cdots = x_1 x_2 \cdots x_{2s-1} x_{2s+1}^2 = x_1 x_2 \cdots x_{2s-2} x_{2s}^2 x_{2s+1} = -x_1 \cdots x_{2s+1},$$

which leads to the claimed formula for $c_{2s} \cdot c_1$. The case of the Chern numbers is analogous. □

Using Proposition 5.1, any top intersection product on the smooth $(2s+1)$-dimensional variety $W^{2s}_{2s^2+2s+1}(X)$ reduces to a sum of monomials in the variables x_i and θ. Next we record the values of these monomials. All terms are essentially reduced to expressions involving $x_1 \cdots x_{2s+1} = C_{2s+1}$.

Proposition 5.2. *Keep the notation from above. The following hold in* $H^{4s+2}\big(W^{2s}_{2s^2+2s+1}(X), \mathbb{Z}\big)$*:*

$$\begin{aligned}
x_1 x_2 \cdots x_{2s+1} &= C_{2s+1},\\
x_1^2 x_2^2 x_3 \cdots x_{2s-1} &= \frac{s(s-1)(s+1)^2(2s+1)^2}{3s(3s+1)} C_{2s+1},\\
x_1^2 x_2 \cdots x_{2s} &= \frac{4s(s+1)}{3s+1} C_{2s+1},\\
x_1^3 x_2 x_3 \cdots x_{2s-1} &= \frac{s^2(s+1)^2(2s-1)(2s+3)}{(3s+1)(3s+2)} C_{2s+1},\\
x_1 x_2 \cdots x_{2s} \cdot \theta &= (2s+1)s C_{2s+1},\\
x_1^2 x_2 \cdots x_{2r-1} \cdot \theta &= \frac{(s+1)^2(2s-1)}{3s+1} x_1 x_2 \cdots x_{2s} \cdot \theta,\\
x_1^2 x_2^2 x_3 \cdots x_{2s-2} &= \frac{(2s-3)(2s+1)(s+1)^2(s+2)}{9(3s+1)} x_1 x_2 \cdots x_{2s} \cdot \theta,\\
x_1^3 x_2 x_3 \cdots x_{2s-2} \cdot \theta &= \frac{(s-1)(s+1)^2(s+2)(2s-1)(2s+3)}{3(3s+1)(3s+2)} x_1 x_2 \cdots x_{2s} \cdot \theta,\\
x_1 x_2 \cdots x_{2s-1} \cdot \theta^2 &= 2s(s+1)(2s+1) C_{2s+1},\\
x_1^2 x_2 \cdots x_{2s-2} \cdot \theta^2 &= \frac{4(s+1)(s-1)(s+2)}{3(3s+1)} x_1 x_2 \cdots x_{2s-1} \cdot \theta^2\\
x_1 x_2 \cdots x_{2s-2} \cdot \theta^3 &= \frac{(2s+1)(2s-1)(s+2)(s+1)s^2}{3} C_{2s+1}.
\end{aligned}$$

We record the formulas for the classes of Z and Y, the proofs being analogous to those of Proposition 4.2.

Proposition 5.3. *Let* $[X, q] \in \mathcal{M}_{2s^2+s,1}$ *be a general pointed curve. If* $c_i := c_i(\mathcal{M}^\vee)$ *are the Chern classes of the tautological vector bundle over* $W^{2s}_{2s^2+2s+1}(X)$*, then one has:*

(1) $[Z] = \pi_2^*(c_{2s}) - 6\pi_2^*(c_{2s-2})\eta\theta + 2\big(s(4s+3)\eta + \gamma\big)\pi_2^*(c_{2s-1}) \in H^{4s}(X \times W^r_d(X), \mathbb{Z})$.
(2) $[Y] = \pi_2^*(c_{2s}) - 2\pi_2^*(c_{2s-2})\eta\theta + \big(2s(s+1)\eta + \gamma\big)\pi_2^*(c_{2s-1}) \in H^{4s}(X \times W^r_d(X), \mathbb{Z})$.

Remark 5.4. For future reference we also record the following formulas, where we recall that $J_1(\mathcal{P})$ denotes the jet bundle of the Poincaré bundle over $X \times \operatorname{Pic}^{2s^2+2s+1}(X)$.

$$c_{2s+1}\big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\big) = \pi_2^*(c_{2s+1}) - 6\pi_2^*(c_{2s-1})\eta\theta + 2\big(s(4s+3)\eta + \gamma\big)\pi_2^*(c_{2s}), \tag{24}$$

$$c_{2s+1}\big(\pi_2^*(\mathcal{M})^\vee - \mathcal{B}^\vee\big) = \pi_2^*(c_{2s+1}) - 2\pi_2^*(c_{2s-1})\eta\theta + \big(2s(s+1)\eta + \gamma\big)\pi_2^*(c_{2s}). \tag{25}$$

The following formulas follow in an analogous way to Proposition 4.3.

Proposition 5.5. *Let X be a general curve of genus $2s^2+s$, let $q \in X$ be a fixed point, and consider the vector bundles $\mathcal{A}_2$ and $\mathcal{B}_2$ on $X \times \mathrm{Pic}^{2s^2+2s+1}(X)$ having fibers*

$$\mathcal{A}_2(y,L) = H^0\big(X, L^{\otimes 2}(-2y)\big) \quad and \quad \mathcal{B}_2(y,L) = H^0\big(X, L^{\otimes 2}(-y-q)\big),$$

respectively. One then has the following formulas:

$$c_1(\mathcal{A}_2) = -4\theta - 4\gamma - 2(3s+1)(2s+1)\eta, \qquad c_1(\mathcal{B}_2) = -4\theta - 2\gamma - (2s+1)^2\eta,$$
$$c_2(\mathcal{A}_2) = 8\theta^2 + 8(6s^2+5s-2)\eta\theta + 16\gamma\theta, \qquad c_2(\mathcal{B}_2) = 8\theta^2 + 4(4s^2+4s-1)\eta\theta + 8\theta\gamma.$$

5.2. **The slope computation.** We are now in a position to complete the proof of Theorem 1.7. Recall that $g = 2s^2+s+1$ and we express the virtual class

$$[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}} = \sigma_*\big(c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\big) = a\lambda - b_0\delta_0 - b_1\delta_1 \in CH^1(\widetilde{\mathcal{M}}_g).$$

The determination of the coefficients a, b_0, and b_1 is similar to the computations for $g = 23$, and we shall highlight the differences. Recall that C_{2s+1} denotes the number of linear series of type $\mathfrak{g}^{2s}_{2s^2+2s}$ on a general curve of genus $2s^2+s$.

Theorem 5.6. *Let X be a general curve of genus $2s^2+s$ and denote by $F_1 \subseteq \widetilde{\Delta}_1 \subseteq \widetilde{\mathcal{M}}_{2s^2+s+1}$ the associated test curve. Then the coefficient of δ_1 in the expansion of $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$ is equal to*

$$b_1 = \frac{1}{2g(X)-2}\sigma^*(F_1)\cdot c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)$$
$$= C_{2s+1}\frac{2s(s-1)(2s+1)}{(2s-1)(3s+1)(3s+2)}\Big(24s^6 - 40s^5 + 18s^4 + 26s^3 + 30s^2 + 47s + 18\Big).$$

Proof. Recall that $W^{2s}_{2s^2+2s+1}(X)$ is a smooth variety of dimension $2s+1$. We work on the product $X \times W^{2s}_{2s^2+2s+1}(X)$ and intersect the degeneracy locus of the map $\phi\colon \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with the surface $\sigma^*(F_1)$, containing Z as an irreducible component. It follows from Proposition 3.12 that Z is the only component contributing to this intersection product, that is,

$$(26) \quad \sigma^*(F_1)\cdot c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) = c_2(\mathcal{F}_{|Z}) - c_2(\mathrm{Sym}^2\mathcal{E}_{|Z}) - c_1(\mathcal{F}_{|Z})c_1(\mathrm{Sym}^2\mathcal{E}_{|Z}) + c_1^2(\mathrm{Sym}^2\mathcal{E}_{|Z}).$$

The kernel $\mathrm{Ker}(\zeta)$ of the vector bundle morphism $\zeta\colon J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$, defined in the proof of Proposition 4.2, is a line bundle on Z. If U is the line bundle on Z with fiber

$$U(y,L) = \frac{H^0(X,L)}{H^0(X,L(-2y))} \hookrightarrow H^0(X, L\otimes \mathcal{O}_{2y})$$

over a point $(y,L) \in Z$, then one has the following exact sequence over Z

$$0 \longrightarrow U \longrightarrow J_1(\mathcal{P}) \longrightarrow \big(\mathrm{Ker}(\zeta)\big)^\vee \longrightarrow 0.$$

From this sequence, it follows that $c_1(U) = 2\gamma + 2(4s^2+3s)\eta + c_1(\mathrm{Ker}(\zeta))$, where the products of $c_1\big(\mathrm{Ker}(\zeta)\big)$ with arbitrary classes ξ coming from $X \times W^{2s}_{2s^2+2s+1}(X)$ can be computed using once more the Harris-Tu formula [HT84]:

$$(27) \qquad c_1\big(\mathrm{Ker}(\zeta)\big)\cdot \xi_{|Z} = -c_{2s+1}\big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\big)\cdot \xi_{|Z}.$$

The Chern classes on the righthand side of (27) have been evaluated in the formula (24). Using a local analysis identical to the one in Theorem 4.4, we conclude that the restriction $\mathcal{F}_{|Z}$ appears in the following exact sequence of vector bundles

$$0 \longrightarrow \mathcal{A}_{2|Z} \longrightarrow \mathcal{F}_{|Z} \longrightarrow U^{\otimes 2} \longrightarrow 0.$$

We obtain the following intersection product on the surface Z:

$$c_2\big(\mathcal{F}_{|Z} - \mathrm{Sym}^2(\mathcal{E})_{|Z}\big) = c_2\big(\mathcal{A}_{2|Z}\big) + 2c_1(\mathcal{A}_{2|Z})\cdot c_1\big(J_1(\mathcal{P})\big) - (2s+3)c_2(\pi_2^*\mathcal{M}^\vee)$$
$$+ \big((2s+2)^2 - s(2s+3)\big)c_1^2\big(\pi_2^*\mathcal{M}^\vee\big) + (2s+2)c_1\big(\mathcal{A}_{2|Z}\big)\cdot c_1\big(\pi_2^*\mathcal{M}^\vee\big)$$
$$+ c_1\big(\mathrm{Ker}(\zeta)\big)\cdot\Big(2c_1(\mathcal{A}_{2|Z}) - 2(r+2)c_1(\pi_2^*\mathcal{M}^\vee)\Big).$$

This expression gets multiplied with the class $[Z]$ computed in Proposition 5.3. The Chern classes of $\mathcal{A}_{2|Z}$ have been computed in Proposition 5.5. We obtain a homogeneous polynomial of degree $2s+2$ on $H^{\mathrm{top}}\big(X\times W^{2s}_{2s^2+2s+1}(X),\mathbb{Z}\big)$. We first consider the terms that *do not* involve $c_1\big(\mathrm{Ker}(\zeta)\big)$. Since $W^{2s}_{2s^2+2s+1}(X)$ is $(2s+1)$-dimensional, each non-zero term in this polynomial has to contain the class η. We collect these terms and obtain the following contribution:

$$2\eta\pi_2^*\Big(-24\ c_{2s-2}\theta^3-8s(s+1)(4s+3)\ c_{2s-1}c_1\theta-(6s^2+15s+12)\ c_{2s-2}c_1^2\theta$$
$$+8s(4s+3)c_{2s-1}\theta^2+3(2s+3)\ c_{2s-2}c_2\theta+s(4s+3)(2s^2+5s+4)\ c_{2s-1}c_1^2$$
$$-s(2s+3)(4s+3)\ c_{2s-1}c_2+24(s+1)\ c_{2s-2}c_1\theta^2-2(2s-1)(s+1)^2\ c_{2s}c_1$$
$$-(8s^2+4s-8)\ c_{2s}\theta\Big)\in H^{\mathrm{top}}\big(X\times W^{2s}_{2s^2+2s+1}(X),\mathbb{Z}\big).$$

Finally, the contribution of the terms containing $c_1\big(\mathrm{Ker}(\zeta)\big)$ is evaluated using (24). The coefficient of $c_1\big(\mathrm{Ker}(\zeta)\big)$ is given by $-2c_1(\mathcal{A}_2)-4(s+1)\pi_2^*(c_1)$. Substituting this for the class ξ in formula (27), we obtain the following contribution to the product $\sigma^*(F_1)\cdot c_2\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)$:

$$4\eta\pi_2^*\Big((3s+1)(2s+1)c_{2s+1}-12\ c_{2s-1}\theta^2+6(s+1)\ c_{2s-1}c_1\theta$$
$$+(16s^2+12s-8)c_{2s}\theta-2s(s+1)(4s+3)c_{2s}c_1\Big)\in H^{\mathrm{top}}\big(X\times W^{2s}_{2s^2+2s+1}(X),\mathbb{Z}\big).$$

The resulting intersection product is then evaluated with *Maple*. Dropping the class η from the sum of the two displayed contributions, one is led to a sum of top Chern numbers on $W^{2s}_{2s^2+2s+1}(X)$, which can be evaluated individually using Propositions 5.1 and 5.2. □

Theorem 5.7. *Let $[X,q]$ be a general pointed curve of genus $2s^2+s$ and let $F_0\subseteq\widetilde{\Delta}_0\subseteq\widetilde{\mathcal{M}}_g$ be the associated test curve. Then the coefficient of δ_0 in the expression of $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$ is equal to*

$$b_0=\frac{\sigma^*(F_0)\cdot c_2\Big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\Big)+b_1}{2s(2s+1)}$$
$$=C_{2s+1}\frac{2(s-1)(24s^8-28s^7+22s^6-5s^5+43s^4+112s^3+100s^2+50s+12)}{9(2s-1)(3s+1)(3s+2)}.$$

Proof. Using Proposition 3.11, we observe that

$$c_2\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)_{|\sigma^*(F_0)}=c_2\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)_{|Y}. \tag{28}$$

To determine the Chern classes of $\mathcal{F}_{|Y}$, we introduce the line bundle V on Y with fiber

$$V(y,L)=\frac{H^0(X,L)}{H^0(X,L(-y-q))}\hookrightarrow H^0(X,L\otimes\mathcal{O}_{y+q})$$

over a point $(y,L)\in Y$. There is an exact sequence of vector bundles over Y

$$0\longrightarrow V\longrightarrow\mathcal{B}\longrightarrow\big(\mathrm{Ker}(\chi)\big)^{\vee}\longrightarrow 0,$$

where the morphism $\chi\colon\mathcal{B}^{\vee}\to\pi_2^*(\mathcal{M})^{\vee}$ was defined in the second part of the proof of Proposition 5.3. Recalling the vector bundle $\mathcal{B}_2$ defined in Proposition 5.5, a local analysis similar to that in the proof of Theorem 4.5 shows that one has an exact sequence on Y

$$0\longrightarrow\mathcal{B}_{2|Y}\longrightarrow\mathcal{F}_{|Y}\longrightarrow V^{\otimes 2}\longrightarrow 0.$$

This determines $c_i(F_{|Y})$, by also using that $c_1\big(\mathrm{Ker}(\chi)\big)=-c_{2s+1}\big(\pi_2^*(\mathcal{M}^{\vee})-\mathcal{B}^{\vee}\big)$, where the right-hand side is estimated using (25).

We first collect terms that do not contain $c_1\big(\mathrm{Ker}(\chi)\big)$, and we obtain the following intersection:

$$\eta\pi_2^*\Big(- 16\ c_{2s-2}\theta^3 - 16s(s+1)^2\ c_{2s-1}c_1\theta - (4s^2+10s+8)\ c_{2s-2}c_1^2\theta + 16s(s+1)c_{2s-1}\theta^2 + (4s+6)\ c_{2s-2}c_2\theta + 2s(s+1)(2s^2+5s+4)\ c_{2s-1}c_1^2 - 2s(s+1)(2s+3)\ c_{2s-1}c_2 + 16(s+1)\ c_{2s-2}c_1\theta^2 - 2(s+1)\ c_{2s}c_1 + 4\ c_{2s}\theta\Big) \in H^{\mathrm{top}}\big(X\times W^{2s}_{2s^2+2s+1}(X),\mathbb{Z}\big).$$

We collect terms containing $c_1\big(\mathrm{Ker}(\chi)\big)$, and obtain an expression in $H^{\mathrm{top}}\big(X\times W^{2s}_{2s^2+2s+1}(X),\mathbb{Z}\big)$ that contributes towards $\sigma^*(F_0)\cdot c_2(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E}))$:

$$\eta\pi_2^*\Big(-16\ c_{2s-1}\theta^2 + 8(s+1)\ c_{2s-1}c_1\theta + 2(2s+1)^2c_{2s+1} + (16s^2+16s-8)c_{2s}\theta - 8s(s+1)^2c_{2s}c_1\Big).$$

We now substitute in (28), and as in the proof of Theorem 5.6, after manipulations we obtain a polynomial of degree $2s+1$ on $W^{2s}_{2s^2+2s+1}(X)$, that we compute by applying (5.1) and (5.2). □

We can now complete the calculation of the slope of $[\widetilde{\mathfrak{D}}_g]^{\mathrm{virt}}$.

Proof of Theorem 1.7. We denote once more by $F_{\mathrm{ell}}\subseteq\widetilde{\mathcal{M}}_g$ the pencil obtained by attaching at the fixed point of a general curve X of genus $2s^2+s$ a pencil of plane cubics at one of the base points of the pencil. Then one has the relation

$$a - 12b_0 + b_1 = F_{\mathrm{ell}}\cdot\sigma_*c_2\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big) = 0.$$

We therefore find the following expression for the λ-coefficient

$$a = \frac{2(s-1)(48s^8-56s^7+92s^6-90s^5+86s^4+324s^3+317s^2+182s+48)}{3(3s+2)(2s-1)(3s+1)}.$$

□

In particular, when $s=3$, we obtain the formula for the class of $[\widetilde{\mathfrak{D}}_{22}]^{\mathrm{virt}}$ in Theorem 1.2.

6. Tropicalizations of linear series

We continue to work over an algebraically closed field K of characteristic zero. For the remainder of the paper, as in §2.4, we choose the field to be spherically complete with respect to a surjective valuation $\nu\colon K^\times\to\mathbb{R}$. Let $R\subseteq K$ be the valuation ring, and κ its residue field. We begin by discussing properties of tropicalizations of not necessarily complete linear series, when the skeleton is an arbitrary tropical curve. In the spirit of [AB15], we also consider the reductions of these linear series, which are themselves linear series on curves over κ.

6.1. **Tropicalizations and reductions of linear series.** Let X be a curve over K with a skeleton $\Gamma\subseteq X^{\mathrm{an}}$. Let D_X be a divisor on X, with $V\subseteq H^0(X,\mathcal{O}(D_X))$ a linear series of rank r. We consider

$$\Sigma = \mathrm{trop}(V) := \{\mathrm{trop}(f)\in \mathrm{PL}(\Gamma) : f\in V\smallsetminus\{0\}\}.$$

Note that $\Sigma\subseteq\mathrm{PL}(\Gamma)$ is a tropical submodule, closed under scalar addition and pointwise minimum.

A *tangent vector* in Γ is a germ of a directed edge. Given a function $\psi\in\mathrm{PL}(\Gamma)$, we write $s_\zeta(\psi)$ for the slope of ψ along a tangent vector ζ.

Lemma 6.1. *For each tangent vector ζ in Γ there are exactly $r+1$ different slopes $s_\zeta(\psi)$, as ψ ranges over Σ.*

Proof. Suppose ζ is based at the point $v\in\Gamma$. We may assume that ζ is an outgoing tangent vector. Then, by the Poincaré-Lelong slope formula [BPR13, Theorem 5.15(3)], the slope $s_\zeta(\mathrm{trop}(f))$ is equal to the order of vanishing of the reduction f_v at the point in $X_v(\kappa)$ corresponding to this tangent direction. Since the reductions form a vector space of dimension $r+1$ over κ [AB15, Lemma 4.3], they have exactly $r+1$ different orders of vanishing. □

Definition 6.2. We write

$$s_\zeta(\Sigma) := (s_\zeta[0], \ldots, s_\zeta[r])$$

for the vector of slopes $s_\zeta(\psi)$, for $\psi \in \Sigma$, ordered so that $s_\zeta[0] < \cdots < s_\zeta[r]$.

Remark 6.3. If $D = \mathrm{Trop}(D_X)$ then the tropical complete linear series $R(D)$ often has far more than $r(D) + 1$ slopes along some tangent vectors. In such cases, Lemma 6.1 shows that the tropicalization of the complete algebraic linear series $H^0(X, \mathcal{O}(D_X))$ is properly contained in $R(D)$.

Given any two tangent vectors, there is a function in Σ with complementary lower and upper bounds on its slopes in these directions, as follows.

Lemma 6.4. *For any pair of tangent vectors ζ and ζ', and for any i, there is a function $\psi \in \Sigma$ such that*

$$s_\zeta(\psi) \leq s_\zeta[i] \quad \text{and} \quad s_{\zeta'}(\psi) \geq s_{\zeta'}[i].$$

Proof. Let $f_0, \ldots, f_r \in V$ be functions satisfying $s_\zeta(\mathrm{trop}(f_i)) = s_\zeta[i]$. Because the functions $\mathrm{trop}(f_i)$ have distinct slopes, the reductions of f_i have distinct orders of vanishing at the point corresponding to ζ, and are therefore linearly independent over κ. It follows that the set of functions $\{f_0, \ldots, f_i\}$ is linearly independent over K and spans a linear subseries V' of rank i. The slopes $s_{\zeta'}(\mathrm{trop}(f))$ for $f \in V'$ therefore take on $i+1$ distinct values in $\{s_{\zeta'}[0], \ldots, s_{\zeta'}[r]\}$, and some f in V' satisfies $s_{\zeta'}(\mathrm{trop}(f)) \geq s_{\zeta'}[i]$, as required. □

The following proposition is not used in the proofs of our main theorems, but the statement is so natural, especially in view of [BN07], that it must be seen as one of the fundamental properties of the tropicalization of a linear series.

Proposition 6.5. *For any effective divisor E on Γ of degree r, there is some $\psi \in \Sigma$ such that $\mathrm{div}(\psi) + D - E$ is effective.*

Proof. We follow the standard argument showing that the rank of the tropicalization of a divisor is greater than or equal to its rank on the algebraic curve, from [Bak08]. Let E_X be an effective divisor of degree r on X that specializes to E. Since V has rank r, there is a function $f \in V$ such that $\mathrm{div}(f) + D_X - E_X$ is effective. Setting $\psi = \mathrm{trop}(f)$ yields the result. □

6.2. **Main strategy for proving the Strong Maximal Rank Conjecture.** Our approach to proving Theorem 1.3 is based on tropical independence and Theorem 1.6, as follows. Let D_X be a divisor of degree $24 + \rho$ and rank 6 on a curve X with skeleton Γ. We set $L := \mathcal{O}(D_X)$, $V := H^0(X, L)$, and $\Sigma := \mathrm{trop}(V)$. Then $\mathrm{Sym}^2 V$ has dimension $\binom{8}{2} = 28$, and any function of the form $\psi + \psi'$, with both ψ and ψ' in Σ, is in the tropicalization of $\mathrm{Im}(\phi_L) \subseteq H^0(X, L^{\otimes 2})$. Therefore, to show that ϕ_L is injective, it suffices to give an independence among 28 pairwise sums of functions in Σ.

Although not logically necessary for the proofs of our theorems, we include the following interpretation of tropical independence in the language of algebraic geometry.

Proposition 6.6. *Let X be a curve over K with skeleton Γ. Let $f_0, \ldots, f_r$ be sections of $\mathcal{O}(D_X)$. Then the following are equivalent:*

1. *The collection $\{\mathrm{trop}(f_0), \ldots, \mathrm{trop}(f_r)\}$ is tropically independent.*
2. *There is a semistable model $\mathcal{X}$ of X, a line bundle $\mathcal{L}$ extending $\mathcal{O}(D_X)$, irreducible components $X_0, \ldots, X_r$ in the special fiber of $\mathcal{X}$, and scalars $a_0, \ldots, a_r \in K$ such that $a_i f_i$ extends to a regular section of $\mathcal{L}$ and vanishes on X_j if and only if $i \neq j$.*

Proof. Suppose $\{\mathrm{trop}(f_0), \ldots, \mathrm{trop}(f_r)\}$ is tropically independent. Then there are real numbers $c_0, \ldots, c_r$ such that $\theta = \min\{\mathrm{trop}(f_i) + c_i\}$ is independent. Choose points $v_0, \ldots, v_r$ in Γ such that $\mathrm{trop}(f_i)$ achieves the minimum uniquely at v_i. Let $\mathcal{X}$ be the model corresponding to some semistable vertex set for Γ that contains $\{v_0, \ldots, v_r\}$ and the tropicalization of every point in the support of D_X, and let X_i be the irreducible component of the special fiber corresponding to v_i.

We define a subsheaf $\mathcal{L}$ of $K(X)$ on $\mathcal{X}$, extending $\mathcal{O}(D_X)$, as follows. A rational function f is a regular section of $\mathcal{L}$ at a point x in the special fiber if and only if there is an affine open neighborhood U of x in the special fiber such that

(i) $\mathrm{div}(f) + D_X$ is effective on $\mathrm{sp}^{-1}(U)$, and
(ii) $\mathrm{trop}(f) \geq \theta$ on $\mathrm{trop}(\mathrm{sp}^{-1}(U))$.

The choice of $\mathcal{X}$, which depends on both D_X and θ, guarantees that this sheaf is locally free of rank 1. Furthermore, by construction, a section f of $\mathcal{O}(D_X)$ is regular on X_i (resp. vanishes on X_i) if and only if $\mathrm{trop}(f) \geq \theta(v_i)$ (resp. $\mathrm{trop}(f) > \theta(v_i)$). In particular, if we choose scalars $a_i \in K^*$ such that $\mathrm{val}(a_i) = c_i$, then the sections $a_i f_i$ of $\mathcal{O}(D_X)$ extend to regular sections of $\mathcal{L}$, and $a_i f_i$ vanishes on X_j if and only if $i \neq j$, as required.

For the converse, given scalars $a_0, \ldots, a_r$, irreducible components $X_0, \ldots, X_r$, and an extension $\mathcal{L}$ of $\mathcal{O}(D_X)$ satisfying (2), set $c_i = \mathrm{val}(a_i)$. By comparing $\mathrm{trop}(f_i)$ with the valuation of a local generator for $\mathcal{L}$ at the generic point of X_i, we conclude that $\mathrm{trop}(f_i) + c_i$ is strictly less than $\mathrm{trop}(f_j) + c_j$ at v_i, and hence $\theta = \min\{\mathrm{trop}(f_i) + c_i\}$ is an independence. □

Remark 6.7. Proposition 6.6 suggests some resemblance between our approach to proving linear independence of sections via tropical independences and the technique used to prove cases of the maximal rank conjecture via limit linear series and linked Grassmannians on chains of elliptic curves in [LOTiBZ17]. Osserman has also developed a notion of limit linear series for curves of pseudocompact type [Oss19b], a class of curves that includes the semistable reduction of the curve X we study here. Relations to the Amini–Baker notion of limit linear series in tropical and nonarchimedean geometry are spelled out in [Oss19a].

6.3. **The tropicalization of a pencil.** The problem of understanding tropicalizations of arbitrary linear series seems hopeless. In most cases that we can analyze, either the behavior is sufficiently similar to the vertex avoiding case, or the essential difference can be explained by the tropicalization of some pencil in the linear series.

We now study the tropicalization of a pencil in a linear series of degree 2 on $\mathbf{P}^1$. This is the simplest possible example that is not completely trivial, yet it illustrates the features that will be essential for our purposes.

Example 6.8. Let Γ be an interval with left endpoint w and right endpoint v, viewed as a skeleton of $X = \mathbf{P}^1$. Let D_X be a divisor specializing to $D = 2w$, let $V \subseteq H^0(X, \mathcal{O}(D_X))$ be a rank 1 linear subseries, and let $\Sigma = \mathrm{trop}(V)$. For each rightward pointing tangent vector ζ, we consider the vector of slopes $s_\zeta = (s_\zeta[0], s_\zeta[1])$. Similarly, we write $s_w = (s_w[0], s_w[1])$ and $s_v = (s_v[0], s_v[1])$ for the rightward outgoing and incoming slopes of functions in Σ at the endpoints w and v, respectively.

Note that $R(D)$ consists of all PL functions on Γ whose rightward slopes are nonincreasing and bounded between 0 and 2. The tropicalization map from $H^0(X, \mathcal{O}(D_X))$ to $R(D)$ is surjective in this case. Now we classify the possibilities for the tropicalization Σ of the pencil.

At each rightward tangent vector ζ, s_ζ is either $(1, 2)$, $(0, 2)$, or $(0, 1)$. We divide the interval Γ into regions, according to these three possibilities. Because the slopes of functions in $R(D)$ do not increase from left to right, the region where s_ζ is equal to $(1, 2)$ must be to the left of the region where it is equal to $(0, 2)$, which in turn must be to the left of the region where it is equal to $(0, 1)$.

Case 1: First, consider the case where there is a non-empty region where s_ζ is equal to $(0, 2)$. (Note that either or both of the regions where s_ζ is equal to $(1, 2)$ and $(0, 1)$ may be empty.)

Choose functions ψ_0 and ψ_1 in Σ such that

$$s_w(\psi_0) = s_w[0] \quad \text{and} \quad s_v(\psi_1) = s_v[1].$$

Because the slopes of functions in $R(D)$ cannot increase from left to right, $s_v(\psi_1)$ must be positive at all rightward tangent vectors. It follows that

$$s_\zeta(\psi_1) = s_\zeta[1]$$

for all ζ. By a similar argument, $s_\zeta(\psi_0) = s_\zeta[0]$ for all ζ. See Figure 1.

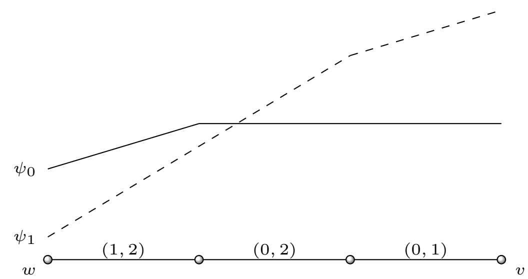

FIGURE 1. The functions ψ_0 and ψ_1, when the $(0, 2)$ region is nonempty.

For any $\psi \in \Sigma$, a similar argument shows that Γ is divided into regions to the left and right where $s_\zeta(\psi) = s_\zeta[1]$ and where $s_\zeta(\psi) = s_\zeta[0]$, respectively. It follows that ψ is a tropical linear combination of ψ_0 and ψ_1. Hence Σ is completely determined by the nonempty $(0, 2)$ region.

Case 2: Suppose the region where $s_\zeta = (0, 2)$ is empty. Then there is a distinguished point $x \in \Gamma$ such that s_ζ is equal to $(1, 2)$ to the left of x and $(0, 1)$ to the right of x. We now consider functions ψ_A and ψ_B such that

$$s_w(\psi_A) = s_w[0] \quad \text{and} \quad s_v(\psi_B) = s_v[1].$$

Note that ψ_A must have slope 1 at all points to the left of x. It continues to have slope 1 for some distance t to the right of x, and then its slope is 0 the rest of the way. Similarly, ψ_B has slope 1 at all points to the right of x, and for some distance t' to the left of x. Then at all points of distance greater than t' to the left of x, the slope of ψ_B is 2.

By taking a tropical linear combination of two functions that agree at x, one with outgoing slope 0 and the other with incoming slope 2, we get a third function $\psi_C \in \Sigma$ with rightward slope 2 everywhere to the left of x and 0 everywhere to the right of x. See Figure 2.

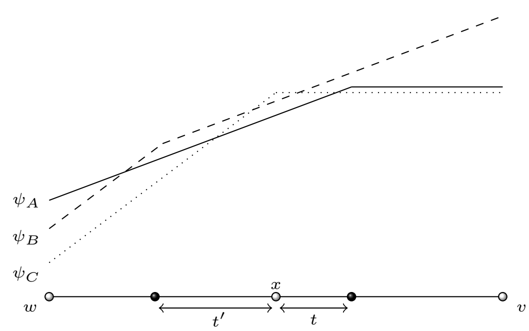

FIGURE 2. The functions ψ_A, ψ_B, and ψ_C when the $(0, 2)$ region is empty.

We claim that the distances t and t' *must be equal.* To see this, note that the functions ψ_A, ψ_B, and ψ_C are tropicalizations of functions in a pencil. Therefore the rational functions in $H^0(X, \mathcal{O}(D_X))$ tropicalizing to these three functions are linearly dependent, and hence ψ_A, ψ_B, and ψ_C *must be tropically dependent.* On each region where all three functions are linear, there are exactly two with the same slope, and this determines the combinatorial type of the tropical dependence, i.e., which functions achieve the minimum on which regions, as shown in Figure 3.

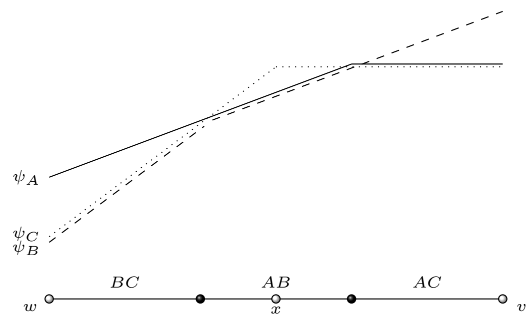

FIGURE 3. The tropical dependence that shows $t = t'$.

Note that all three functions achieve the minimum together at two points: the point of distance t to the right of x, and the point of distance t' to the left of x. Comparing the slopes of ψ_C to those of ψ_A shows that $\psi_C - \psi_A$ is equal to t' at x, and also equal to t. This proves that $t = t'$, as claimed. Analogous arguments show that an arbitrary function $\psi \in \Sigma$ is a tropical linear combination of ψ_A, ψ_B, and ψ_C, so Σ is determined by x and t.

Remark 6.9. Let ψ^∞ be a function that has slope 1 everywhere on Γ. (The notation is meant to suggest that ψ^∞ is the shared limit of ψ_A and ψ_B, as $t \to \infty$.) In Case 2, the functions ψ_A, ψ_B, and ψ_C are all tropical linear combinations of ψ_0, ψ_1, and ψ^∞, regardless of x and t.

In §9, we will give case-by-case arguments that depend on the tropicalization Σ of a linear series. In each case, we identify functions in Σ analogous to ψ_A, ψ_B, and ψ_C that can be expressed in terms of simpler functions in $R(D)$ that are analogous to ψ_0, ψ_1, and ψ^∞. We carry out the most technical steps in our argument in §8 using these simpler functions, which we call *building blocks*, and then explain how to adapt the construction to the particulars of Σ, in each case.

6.4. **Chains of loops.** We now focus on the case where Γ is a chain of g loops with bridges. This graph has $2g + 2$ vertices, labeled $w_0, \ldots, w_g$, and $v_1, \ldots, v_{g+1}$, with bridges connecting w_i to v_{i+1}. There are two edges connecting v_k to w_k, whose lengths are denoted ℓ_k and m_k, as shown in Figure 4.

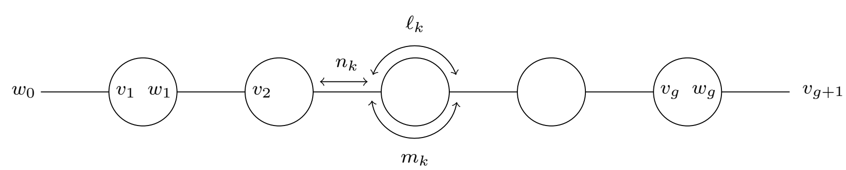

FIGURE 4. The chain of loops Γ.

We write γ_k for the kth loop, formed by the two edges connecting v_k and w_k, for $1 \leq k \leq g$. We write β_k the kth bridge, which connects w_{k-1} and v_k, for $0 \leq k \leq g$, and has length n_k.

Throughout, we assume that Γ has admissible edge lengths in the following sense.

Definition 6.10. The graph Γ has *admissible edge lengths* if

$$\ell_{k+1} \ll m_k \ll \ell_k \ll n_{k+1} \ll n_k \text{ for all } k.$$

Remark 6.11. These conditions on edge lengths are more restrictive than those in [JP16] and [CDPR12]. Our arguments here, e.g., in Lemma 7.21, require not only that the bridges are much longer than the loops, but also that each loop is much larger than the loops that come after it. For illustrative purposes, we will generally draw the loops and bridges as if they are the same size.

6.5. **Special divisors on a chain of loops.** By the Riemann-Roch Theorem, every divisor class of degree d on Γ has rank at least $d-g$ [BN07]. The special divisor classes on Γ, i.e., the classes of degree d and rank strictly greater than $d-g$, are classified in [CDPR12]. We briefly recall the structure of this classification and refer the reader to the original paper for further details.

Every divisor on Γ is equivalent to a unique *break divisor* D, with multiplicity $d-g$ at w_0, and precisely one point of multiplicity 1 on each loop γ_k; see, for instance, [ABKS14]. In this way, $\mathrm{Pic}^d(\Gamma)$ is naturally identified with $\prod_{k=1}^{g}\gamma_k$.

We choose a coordinate on each loop γ_k, as follows. Since the top and bottom edges have lengths ℓ_k and m_k, respectively, the loop has length $\ell_k + m_k$.

Definition 6.12. Let $x_k(D) \in \mathbb{R}/(\ell_k+m_k)\cdot\mathbb{Z}$ be the counterclockwise distance from v_k to the unique point in the support of $D_{|\gamma_k}$.

Then D is determined uniquely by its degree d and the coordinates $(x_1(D),\ldots,x_g(D))$. When D is fixed, we omit it from the notation, and write simply x_k.

The Brill-Noether locus $W^r_d(\Gamma) \subseteq \mathrm{Pic}^d(\Gamma)$ parametrizing divisor classes of degree d and rank r is a union of translates of $\rho(g,r,d)$-dimensional coordinate subtori. These tori are in bijection with standard Young tableaux T on a rectangle of size $(r+1)\times(g-d+r)$, with entries from $\{1,\ldots,g\}$; the coordinates on the subtorus are those x_k such that k does not appear in T.

Definition 6.13. A class $[D] \in W^r_d(\Gamma)$ is *vertex avoiding* if there is a unique divisor $D_i \sim D$ such that $D - iw_0 - (r-i)v_{g+1}$ is effective, for $0 \le i \le r$.

Being in $W^r_d(\Gamma)$ implies that there exists some $D'_i \sim D$ such that $D'_i - iw_0 - (r-i)v_{g+1}$ is effective, so the essential part of the definition is the uniqueness.

An open dense subset of each torus corresponding to a standard Young tableau consists of vertex avoiding divisor classes; those that are not vertex avoiding form a union of subtori of positive codimension. When D is vertex avoiding, we choose $\varphi_i \in \mathrm{PL}(\Gamma)$ such that

$$\mathrm{div}(\varphi_i) = D_i - D.$$

The uniqueness of D_i ensures that φ_i is unique up to an additive constant. In particular, if D_X is a divisor of rank r such that $\mathrm{Trop}(D_X) = D$ then $\varphi_i \in \mathrm{trop}(H^0(X,\mathcal{O}(D_X))$ for $0 \le i \le r$.

Remark 6.14. The combinatorial advantages of working with such functions φ_i (rather than, say, functions ψ_i such that $D_i = D_0 + \mathrm{div}(\psi_i)$) were discovered and used earlier by Pflueger [Pfl17].

6.6. **Slopes along bridges.** The slopes of PL functions along the bridges of Γ, and especially the incoming and outgoing slopes at each loop, play a special role in controlling which functions can achieve the minimum on which regions of the graph in a given tropical linear combination. We use the following notation for these slopes.

Given $\psi \in \mathrm{PL}(\Gamma)$, let $s_k(\psi)$ be the incoming slope from the left at v_k, and let $s'_k(\psi)$ be the outgoing slope of ψ to the right at w_k.

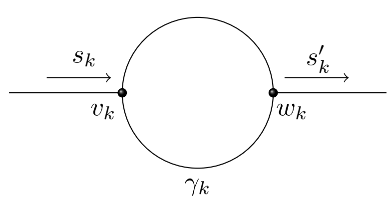

FIGURE 5. The slopes s_k and s'_k.

Note that $s'_{k-1}(\psi)$ and $s_k(\psi)$ are the rightward slopes of ψ at the beginning and end of the kth bridge. In particular, if ψ has constant slope along each bridge then $s_k(\psi)$ is its slope along β_k.

6.7. **Tropicalization of linear series on chains of loops.** In this subsection, we make first steps toward classifying linear series of degree d and rank r on a chain of g loops with admissible edge lengths, extending the combinatorial classification of divisor classes of degree d and rank r in [CDPR12].

6.7.1. *Slope vectors and ramification.* By Lemma 6.1, for each tangent vector in Γ, there are exactly $r+1$ distinct slopes of functions in $\Sigma = \mathrm{trop}(V)$. The incoming and outgoing (rightward) slopes at each loop play a special role in our analysis. We write

$$s_k(\Sigma) := (s_k[0], \ldots, s_k[r]) \quad \text{and} \quad s'_k(\Sigma) := (s'_k[0], \ldots, s'_k[r])$$

for the $r+1$ slopes that occur as $s_k(\varphi)$ and $s'_k(\varphi)$, respectively, for $\varphi \in \Sigma$, written in increasing order. The changes in these slope vectors, as one moves from left to right across the loops and bridges, are bounded as follows, in terms of the coordinates $(x_1, \ldots, x_g)$ given by Definition 6.12.

Proposition 6.15. *The differences between incoming and outgoing slopes at γ_k are bounded by*

$$s'_k[i] - s_k[i] \leq \begin{cases} 1 & \text{if } x_k \equiv (s_k[i]+1)m_k \pmod{\ell_k + m_k} \\ & \text{and } s_k[i+1] \neq s_k[i]+1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the differences between incoming and outgoing slopes at β_k are bounded by

$$s_k[i] - s'_{k-1}[i] \leq 0.$$

Proof. To prove the inequality for γ_k, first note that, by Lemma 6.4, there is a function $\varphi_i \in \Sigma$ such that $s_k(\varphi_i) \leq s_k[i]$ and $s'_k(\varphi_i) \geq s'_k[i]$. It now suffices to show that $s'_k(\varphi_i) - s_k(\varphi_i)$ is bounded by 1 if $x_k = (s_k(i)+1)m_k \bmod (\ell_k + m_k)$, and by 0 otherwise. The fact that these bounds hold for any function in $R(D)$ is the essential content of [CDPR12, Example 2.1].

To prove the inequality for bridges, choose $\varphi_i \in \Sigma$ such that $s_k(\varphi_i) \geq s_k[i]$ and $s'_{k-1}(\varphi_i) \leq s'_{k-1}[i]$. Since the support of D is disjoint from the interior of the bridge β_k, the slope of φ_i cannot increase along the bridge, and hence $s_k(\varphi_i) \leq s'_{k-1}(\varphi_i)$, as required. □

By Proposition 6.15, for a fixed k we have $s'_k[i] - s_k[i] \leq 1$, with equality for at most one i. We define the *multiplicity* of the kth loop to be the total amount by which the coordinatewise difference $s'_k(\Sigma) - s_k(\Sigma)$ deviates from this bound, and similarly for the kth bridge β_k.

Definition 6.16. The multiplicity of γ_k and β_k are defined, respectively, to be

$$\mu(\gamma_k) := 1 - \sum_{i=0}^{r} (s'_k[i] - s_k[i]) \quad \text{and} \quad \mu(\beta_k) := -\sum_{i=0}^{r} \left(s_k[i] - s'_{k-1}[i]\right).$$

By Proposition 6.15, the multiplicity of each loop or bridge is nonnegative.

The multiplicities of loops and bridges record where, and by how much, the rightward slopes of Σ fail to increase or stay the same, as expected. We also keep track of the extent to which the rightward slopes of Σ may be lower than expected at the leftmost point w_0, or higher than expected at the rightmost point v_{g+1}. For a vertex avoiding divisor of degree d and rank r, the rightward slopes at w_0 are $(d-g-r, d-g-r+1, \ldots, d-g)$, and the rightward slopes at v_{g+1} are $(0, \ldots, r)$.

Definition 6.17. The *ramification weights* of Σ at w_0 and v_{g+1} are

$$\mathrm{wt}(w_0) := \sum_{i=0}^{r} (d - g - r + i - s'_0[i]) \quad \text{and} \quad \mathrm{wt}(v_{g+1}) := \sum_{i=0}^{r} (s_{g+1}[i] - i).$$

Our choice of terminology reflects the fact that these are the ramification weights of the reductions of V to $\kappa(X_{w_0})$ and $\kappa(X_{v_{g+1}})$, respectively. Importantly, if $p \in X$ is a point specializing to w_0, then

$$s'_0[i] \leq d - g - a^V_{r-i}(p). \tag{29}$$

It follows that the ramification weight of Σ at w_0 is greater than or equal to that of V at p. Similarly, if $q \in X$ is a point specializing to v_{g+1}, then

$$s_{g+1}[i] \geq a_i^V(q). \tag{30}$$

It follows that the ramification weight of Σ at v_{g+1} is greater than or equal to that of V at q.

The main theorem of [CDPR12] says that the space of divisor classes of degree d and rank r on Γ has dimension $\rho(g, r, d) = g - (r+1)(g - d + r)$. We have the following analogue for tropicalizations of linear series.

Proposition 6.18. *The sum of the multiplicities of all loops and bridges plus the ramification weights at w_0 and v_{g+1} is equal to the Brill-Noether number $\rho = g - (r+1)(g - d + r)$.*

Proof. Starting from the definitions of the multiplicities of the loops and bridges and then collecting and canceling terms, we have

$$\sum_{k=1}^{g} \mu(\gamma_k) + \sum_{k=1}^{g+1} \mu(\beta_k) = g + \sum_{i=0}^{r} (s'_0[i] - s_{g+1}[i]).$$

Moreover,

$$\mathrm{wt}(w_0) = (r+1)(d-g) - \binom{r+1}{2} - \sum_{i=0}^{r} s'_0[i], \quad \text{and} \quad \mathrm{wt}(v_{g+1}) = \sum_{i=0}^{r} s_{g+1}[i] - \binom{r+1}{2}.$$

Adding these together and again collecting and canceling terms gives $g - (r+1)(g - d + r)$. □

In the vertex avoiding case, the lingering loops have multiplicity 1, and all other multiplicities and ramification weights are 0. In general, the distribution of multiplicities and ramification weights is a useful indication of how and where the tropicalization of a given linear series differs essentially from the vertex avoiding case.

6.8. **Switching loops and bridges.** Say that the *slope index* of a function $\psi \in \Sigma$ at a rightward pointing tangent vector ζ is the unique i such that $s_\zeta(\psi) = s_\zeta[i]$. In the vertex avoiding case, the slope index of each $\psi \in \Sigma$ at tangent vectors along bridges is nonincreasing as we move from left to right across the graph. This can fail for more general linear series. An analogous phenomenon for linear series on $\mathbf{P}^1$ is illustrated in Case 2 of Example 6.8; the functions ψ_A and ψ_B switch up from slope index 0 to slope index 1 as we move from left to right past x.

This example illustrates a key subtlety in the behavior of tropicalizations of linear series. Most importantly for our subsequent arguments, such switching behavior is always associated to bridges or loops of positive multiplicity, and the combinatorial possibilities are manageable when these multiplicities are small, as we now explain.

Definition 6.19. A loop γ_k is a *switching loop* if there is some $\varphi \in \Sigma$ such that

$$s_k(\varphi) \leq s_k[h] \text{ and } s'_k(\varphi) \geq s'_k[h+1].$$

Similarly, β_k is a *switching bridge* if there is some $\varphi \in \Sigma$ such that

$$s'_{k-1}(\varphi) \leq s'_{k-1}[h] \text{ and } s_k(\varphi) \geq s_k[h+1].$$

When we wish to emphasize the index h for which such a function φ exists in Σ, we say that γ_k or β_k *switches slope h*. The terminology is chosen to emphasize that, on a switching loop, there is a function φ with slope $s_k[h]$ coming in from the left that "switches up" to having slope $s'_k[h+1]$ going out to the right. Similarly, on a switching bridge, there is a function that switches up from slope $s'_{k-1}[h]$ at the beginning of the bridge to slope $s_k[h+1]$ at the end of the bridge.

6.8.1. *Slope patterns on switching bridges.* The slope of a function in Σ does not increase along any bridge. Therefore, if β_k switches slope h, we must have

$$s_k[h+1] \le s'_{k-1}[h].$$

Since $s'_{k-1}[h] < s'_{k-1}[h+1]$ and $s_k[h] < s_k[h+1]$, any switching bridge has multiplicity at least 2.

Since we focus on the case where $\rho \le 2$, and the sum of all multiplicities of loops and bridges is at most ρ, we will only consider switching bridges of multiplicity exactly 2. If β_k is a bridge of multiplicity 2 that switches slope h, then we must have

$$s'_{k-1}[h+1] - 1 = s'_{k-1}[h] = s_k[h+1] = s_k[h] + 1 \text{ and } s_k[i] = s'_{k-1}[i] \text{ for } i \ne h, h+1.$$

We denote this pattern of slopes for a bridge that switches slope h schematically in Figure 6.

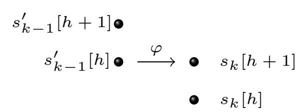

FIGURE 6. Switching pattern on bridges of multiplicity 2.

The heights of the dots represent the values of the slopes, and the arrow indicates that φ has slope $s'_{k-1}[h]$ at the beginning of the bridge β_k and $s_k[h+1]$ at the end of the bridge.

6.8.2. *Slope patterns on switching loops.* Figure 7 gives a similar schematic depiction of the only possible pattern of slopes on a loop of multiplicity 1 that switches slope h. Here, $s_k[i] = s'_k[i]$ for all i, $s_k[h+1] = s_k[h] + 1$, and there is a function $\varphi \in \Sigma$ such that

$$s_k(\varphi) = s_k[h] \text{ and } s'_k(\varphi) = s_k[h+1].$$

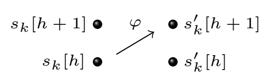

FIGURE 7. Switching pattern on a loop of multiplicity 1.

Note that $s'_k[h] > s_k[h]$ implies that $x_k = (s_k[h]+1)m_k$.

One may similarly classify the patterns of slopes that may appear on switching loops of multiplicity 2. One possibility is that the slopes are just as in Figure 7, except with $s'_k[i] = s_k[i] - 1$ for some $i \ne h, h+1$. The remaining three possibilities are represented schematically in Figure 8.

The relative heights of the dots on the left of each picture represent the relative values of $s_k[h]$ and $s_k[h+1]$, while the dots on the right represent the values of $s'_k[h]$ and $s'_k[h+1]$, just as in Figure 7. (We will omit the labels in such diagrams, when h, k, and φ are clear from context.)

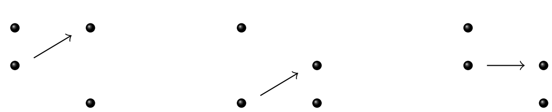

FIGURE 8. Switching patterns on loops of multiplicity at most 2.

In the first two cases, the positive slope of the arrow indicates $s'_k(\varphi) > s_k(\varphi)$, and this implies that $x_k = (s_k[h]+1)m_k$. In each of these cases, $s'_k[i] = s_k[i]$ for all $i \ne h, h+1$. In the remaining case, multiplicity 2 requires that there is a unique $i \ne h, h+1$ such that $s'_k[i] = s_k[i] + 1$, and $s'_k[j] = s_k[j]$ for $j \ne h, h+1, i$.

Lemma 6.20. *If $\mu(\gamma_k) \le 2$, then there is at most one value of h such that γ_k switches slope h.*

Proof. This follows directly from the above classification of switching patterns. □

Remark 6.21. By Proposition 6.18, the sum of the multiplicities of all loops and bridges is at most ρ. When $\rho = 1$, this means that there are no switching bridges and at most one switching loop, as in Figure 7. Moreover, if there is such a switching loop then all other bridges and loops have multiplicity 0. The possibilities when $\rho = 2$ are more complicated, but still manageable (far more so than they would be for $\rho > 2$); indeed, the classification of combinatorial possibilities for switching loops and bridges is the essential place in which we use the assumption $\rho \leq 2$ in the proofs of our main theorems.

6.8.3. *Existence of distinguished functions.* In the vertex avoiding case, there are distinguished functions $\varphi_i \in R(D)$, for all $0 \leq i \leq r$, unique up to adding a constant, such that $s_k(\varphi) = s_k[i]$ and $s'_k(\varphi_i) = s'_k[i]$ for all k, and these functions are always in Σ. We now identify the subset of indices in $\{0, \ldots, r\}$ for which such distinguished functions exist, in the general case.

Lemma 6.22. *For each $0 \leq i \leq r$, either there is a function $\varphi_i \in \Sigma$ such that*

$$s_k(\varphi_i) = s_k[i] \text{ and } s'_k(\varphi_i) = s'_k[i] \text{ for all } k,$$

or there is a loop or bridge that switches slope i or $i - 1$.

Proof. By Lemma 6.4, there is a function $\varphi_i \in \Sigma$ with $s'_0(\varphi_i) \leq s'_0[i]$ and $s_{g+1}(\varphi_i) \geq s_{g+1}[i]$. By induction, if there are no loops or bridges that switch slope i, we see that $s_k(\varphi_i) \leq s_k[i]$ and $s'_k(\varphi_i) \leq s'_k[i]$ for all k. Similarly, if there are no loops or bridges that switch slope $i - 1$, we see that $s_k(\varphi_i) \geq s_k[i]$ and $s'_k(\varphi_i) \geq s'_k[i]$ for all k. □

Corollary 6.23. *If there are no switching loops or switching bridges, then for all i there is a function $\varphi_i \in \Sigma$ such that*

$$s_k(\varphi_i) = s_k[i] \text{ and } s'_k(\varphi_i) = s'_k[i] \text{ for all } k.$$

There do exist cases that are not vertex avoiding that nevertheless have no switching loops or bridges. In these cases, we will produce an independence among the pairwise sums φ_{ij} of the distinguished functions φ_i given by Corollary 6.23 in a manner similar to that of §7. In cases with switching loops and bridges, the analysis requires more careful consideration of the possibilities for Σ. In each case, we identify functions analogous to ψ_A, ψ_B and ψ_C in Example 6.8 which can be used as substitutes in the construction of an independence. These cases occupy most of §9.

7. The vertex avoiding case

We continue our analysis of tropicalizations of linear series on chains of loops, moving toward a proof of the two cases of the Strong Maximal Rank Conjecture stated in Theorem 1.3. In this section, we prove injectivity of ϕ_L in the vertex avoiding case, using tropical independence. From this point onward, we assume that $r = 6$, $g = 21 + \rho$, $d = 24 + \rho$, and $\rho \leq 2$. We will write

$$\varphi_{ij} = \varphi_i + \varphi_j,$$

for the pairwise sums of the distinguished functions $\varphi_0, \ldots, \varphi_6$ in Σ.

Theorem 7.1. *Let D be a break divisor of degree $24 + \rho$ and rank 6 on a chain of $g = 21 + \rho$ loops whose class is vertex avoiding. Then $\{\varphi_{ij} : 0 \leq i, j \leq 6\}$ is tropically independent.*

To prove Theorem 7.1, we give an algorithm that constructs an explicit independence

$$\theta = \min_{ij}\{\varphi_{ij} + c_{ij}\}.$$

Recall that each vertex avoiding class is contained in the torus of special divisor classes corresponding to a unique standard Young tableau $T = T(D)$ of shape $(r + 1) \times (g - d + r)$ with entries from $\{1, \ldots, g\}$. In terms of this tableau, the slope of φ_i along the bridge β_k is

$$s_k(\varphi_i) = i - (g - d + r) + \#\{\text{entries} < k \text{ in column } r + 1 - i\}.$$

A loop γ_k is *lingering* if k is not an entry in the tableau. If γ_k is lingering, then the slopes linger at the kth step, meaning that $s_{k+1}(\varphi_i) = s_k(\varphi_i)$ for all i.

7.1. **An illustrative example.** We begin by proving the theorem for one example, chosen at random. We construct an independence from the 28 functions φ_{ij}, moving from left to right along the graph and applying some rough approximation of a greedy algorithm. The essential content of the remainder of this section is that this method can be made into a precise algorithm that works in all vertex avoiding cases, when $g = 21 + \rho$, $r = 6$, and $d = g + 3$.

In the example below, $g = 22$, and γ_{18} is the unique lingering loop. In the vertex avoiding case, adding lingering loops does not create any significant difficulties. However, there are $692,835$ standard tableaux on a 3×7 rectangle with entries from $\{1, \ldots, 21\}$, so it is important to have one method that works uniformly in all of these cases.

Example 7.2. Let Γ be a chain of 22 loops with admissible edge lengths, and let $[D]$ be a vertex avoiding class of degree 25 and rank 6 associated to the tableau T in Figure 9.

01	03	06	09	10	13	15
02	05	07	12	16	19	20
04	08	11	14	17	21	22

FIGURE 9. A randomly generated 3×7 tableau T.

The independence $\theta = \min_{ij}\{\varphi_{ij} + c_{ij}\}$ that we construct is depicted schematically in Figure 10. The graph should be read from left to right and top to bottom, so the first 7 loops appear in the first row, with γ_1 on the left and γ_7 on the right, and γ_{22} is the last loop in the third row. The 48 dots indicate the support of the divisor $D' = 2D + \mathrm{div}(\theta)$. Note that $\deg(D') = 50$; the points on the bridges β_4 and β_{13} appear with multiplicity 2, as marked. Each of the 28 functions φ_{ij} achieves the minimum uniquely on the connected component of the complement of $\mathrm{Supp}(D')$ labeled ij.

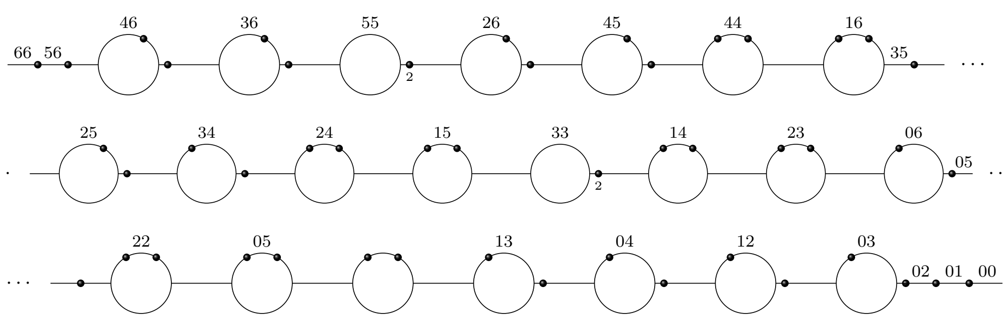

FIGURE 10. The divisor $D' = 2D + \mathrm{div}(\theta)$. The function φ_{ij} achieves the minimum uniquely on the region labeled ij in $\Gamma \smallsetminus \mathrm{Supp}(D')$.

We now explain a procedure for constructing this independence.

First bridge. On β_1, there is a unique function φ_{66} of highest slope (equal to 6). If it does not achieve the minimum at the left endpoint w_0, then it will never achieve the minimum. So we add a constant c_{66} to φ_{66} and assign it to achieve the minimum uniquely on the first third of β_1. Similarly, φ_{56} is the unique function with the next highest slope 5 on β_1. If some other unassigned function is equal to φ_{66} at the midpoint of β_1, then φ_{56} will never achieve the minimum uniquely. So we assign φ_{56} to achieve the minimum on the second third of β_1, and choose c_{56} accordingly.

First loop. At γ_1, there are three unassigned functions with incoming slope at most 4 and outgoing slope at least 4, namely φ_{36}, φ_{46}, and φ_{55}. Of these, φ_{46} has strictly higher slope (equal to 5) on the outgoing bridge, so if φ_{55} achieves the minimum on γ_1 and φ_{46} does not, then φ_{46} will never achieve the minimum anywhere to the right. So we assign φ_{46} to achieve the minimum uniquely on γ_1 and choose c_{46} accordingly.

Loops γ_2 through γ_5. Each of these is similar to γ_1. When we arrive at γ_k, moving from left to right across the graph, there are three unassigned functions with incoming slope at most 4 and outgoing slope at least 4. Exactly one of these has outgoing slope strictly greater than 4. We assign that function to achieve the minimum uniquely on γ_k and choose its coefficient accordingly.

Loops γ_6 and γ_7 On γ_6, there are three functions with incoming slope at most 4 and outgoing slope at least 4. This time, all three have outgoing slope exactly 4. If we were to choose coefficents so that all three are equal at the right endpoint w_6, then an explicit computation shows that φ_{44} achieves the minimum uniquely on a segment in the top half of the loop. We can therefore choose coefficients so that φ_{44} achieves the minimum uniquely on a (slightly smaller) interval in the top half of the loop, and the other two functions achieve the minimum uniquely at w_6.

We apply the same procedure to γ_7. There are now two functions with incoming slope at most 4 and outgoing slope at least 4, and if we choose coefficents so that they agree at w_7, then an explicit computation shows that φ_{16} achieves the minimum uniquely on a segment in the top half of the loop. We can therefore choose coefficients so that φ_{16} achieves the minimum uniquely on a (slightly smaller) interval in the top half of the loop, and φ_{35} achieves the minimum uniquely at w_7.

We now observe that there are no other unassigned functions with slope 4 on γ_8. So we assign φ_{35} to achieve the minimum on the first half of β_8, and begin to consider functions with slope 3.

Loop γ_8. There are four unassigned functions with incoming slope at least 3 and outgoing slope at most 3. Of these, only φ_{25} has outgoing slope strictly greater than 3. So we assign φ_{25} to achieve the minimum uniquely on γ_8 and choose its coefficient accordingly.

Loops γ_9 through γ_{15}. Each of these is handled similarly to earlier loops. There are 3 unassigned functions with incoming slope at most 3 and outgoing slope at least 3 on γ_9 and γ_{10}, and 2 such functions on γ_{11} through γ_{15}. In some cases, exactly one has outgoing slope strictly greater than 3, and we assign it to achieve the minimum. Otherwise, if we were to set coefficients so that they all agree on the right endpoint w_k, then an explicit computation shows that one achieves the minimum uniquely on an interval in the top half of the loop, and we assign that function to achieve the minimum on a (slightly smaller) interval.

The bridge β_{16}. There is only one unassigned function left with slope 3 on β_{16}, namely φ_{05}. We choose its coefficient so that φ_{05} achieves the minimum on the first half of β_{16}, and begin considering functions with slope 2.

Loops γ_{16} through γ_{22}. Each non-lingering loop on this portion of the graph is handled similarly to loops γ_8 through γ_{15}; there are 2 or 3 unassigned functions with incoming slope at most 2 and outgoing slope at least 2, and we assign one to achieve the minimum, by the same methods as previously. On the lingering loop γ_{18}, we do nothing. Whatever function ends up achieving the minimum on the incoming bridge β_{17} will also achieve the minimum on γ_{18} and the outgoing bridge β_{18}. In this example, it is φ_{04}. For the purposes of constructing an independence, we proceed as if the lingering loop were not there at all.

Final bridge. When we arrive at the last bridge β_{22}, there are three remaining unassigned functions, φ_{02}, φ_{01} and φ_{00}, with slopes 2, 1, 0, and we choose coefficients so that each one achieves the minimum uniquely on part of the bridge.

7.2. **Slopes of the independence.** The remainder of this section gives an algorithm that generalizes the construction in Example 7.2 and verifies its validity. The essence of the construction is the same as in the example. We divide the graph into three *blocks*, analogous to the three rows in

Figure 10. Within each block, the slope of θ will be nearly constant on each bridge, equal to 4, 3, and 2 on bridges within the first, second, and third blocks, respectively. By nearly constant, we mean that the slope of θ might be different (1 or 2 higher) for a short distance at the beginning of the bridge, but since the bridges are very long, the average slope over each bridge within a block will be very close to 4, 3, or 2, according to the block. On the bridges between the blocks, the slope decreases by 1 at the midpoint of the bridge.

When we verify the algorithm, one key step is a counting argument, comparing the number of functions that satisfy the appropriate slope condition on each block (incoming slope at most s and outgoing slope at least s) to the number of loops in that block. In order to facilitate this verification, we specify in advance the last loops of the first and second blocks, in terms of the tableau T corresponding to the given vertex avoiding divisor. These will be the loops numbered $z(T)$ and $z'(T)$, characterized as follows.

Definition 7.3. Let $z(T)$ be the 6th smallest entry appearing in the union of the first two rows of T, and choose $z'(T)$ so that $z'(T)+2$ is the 10th smallest entry appearing in the union of the last two rows.

Example 7.4. In Figure 9, the 6th smallest entry in the union of the first two rows is 7, and the 10th smallest entry in the union of the last two rows is 17. So $z(T) = 7$ and $z'(T) = 15$.

When T is fixed, we omit it from the notation, and write simply z and z'. So γ_z is the last loop of the first block, and the slope of the independence θ will drop from 4 to 3 at the midpoint of β_{z+1}. Similarly, $\gamma_{z'}$ is the last loop of the second block and the slope of θ drops from 3 to 2 at the midpoint of $\beta_{z'+1}$. The incoming slope of θ at the loop γ_k is:

$$s_k(\theta) = \begin{cases} 4 & \text{if } k \leq z, \\ 3 & \text{if } z < k \leq z', \\ 2 & \text{if } z' < k \leq g. \end{cases}$$

7.3. **Permissible functions.** The following definition highlights a natural necessary condition for a PL function that is linear on each bridge to achieve the minimum at some point of a given loop, provided that $\theta = \min_\psi\{\psi + c_\psi\}$ has slopes along bridges as specified above.

Definition 7.5. Let $\psi \in \mathrm{PL}(\Gamma)$ be a function with constant slope along each bridge. We say that ψ is *permissible* on γ_k if

(1) $s_j(\psi) \leq s_j(\theta)$ for all $j \leq k$,
(2) $s_{k+1}(\psi) \geq s_k(\theta)$, and
(3) if $s_\ell(\psi) < s_\ell(\theta)$ for some $\ell > k$, then $s_{k'}(\psi) > s_{k'}(\theta)$ for some k' such that $k < k' < \ell$.

Remark 7.6. In the vertex avoiding case, we consider permissibility only for $\psi = \varphi_{ij}$. Nevertheless, we discuss permissibility for arbitrary functions with constant slopes along bridges, since this more general notion will be needed in the general case.

To understand the motivation for the definition, keep in mind that, since Γ has admissible edge lengths in the sense of Definition 6.10, the bridges adjacent to a loop are much longer than the edges in the loop, and both bridges and loops get much smaller as we move from left to right across the graph. Therefore, any ψ with constant slopes along bridges that achieves the minimum on γ_k must have smaller than or equal slope, when compared with the minimum θ, on every bridge to the left and greater than or equal slope on the first half of the bridge immediately to the right. Moreover, if it has smaller slope on a bridge further to the right, then it must have had strictly larger slope on some bridge in between.

In the vertex avoiding case, the condition for φ_{ij} to be permissible on γ_k simplifies as follows.

Lemma 7.7. *In the vertex avoiding case, φ_{ij} is permissible on γ_k if and only if*

$$s_k(\varphi_{ij}) \leq s_k(\theta) \leq s_{k+1}(\varphi_{ij}).$$

Proof. The slopes $s_k(\varphi_{ij})$ are nondecreasing in k, while $s_k(\theta)$ is nonincreasing. □

We also note the following, which holds for any function with constant slopes along bridges, not just those of the form φ_{ij}.

Lemma 7.8. *Let $\psi \in \mathrm{PL}(\Gamma)$ have constant slope along each bridge. Then either*

(1) $s_1(\psi) > s_1(\theta)$, *or*
(2) $s_{g+1}(\psi) < s_g(\theta)$, *or*
(3) *there is a k such that ψ is permissible on γ_k.*

Proof. Suppose that $s_1(\psi) \leq s_1(\theta)$ and $s_{g+1}(\psi) \geq s_g(\theta)$. If $s_k(\psi) \leq s_k(\theta)$ for all k, then ψ is permissible on γ_g. Otherwise, consider the smallest value of k such that $s_k(\psi) > s_k(\theta)$. Then ψ is permissible on γ_{k-1}. □

Remark 7.9. The set of loops on which a given function ψ with constant slope along each bridge is permissible consists of *consecutive* loops. The last loop where a function is permissible is γ_k, where k is the smallest value such that $s_{k+1}(\psi) > s_{k+1}(\theta)$. The first loop where a function is permissible is γ_ℓ, where ℓ is the largest value such that $\ell \leq k$ and $s_\ell(\psi) < s_\ell(\theta)$.

7.4. **Counting permissible functions.** Our algorithm is organized around keeping track of which functions φ_{ij} are permissible at each step, as we move from left to right across the graph. The set of indices k such that φ_{ij} is permissible on γ_k are the integers in an interval, as noted in Remark 7.9, so we pay special attention to the first and last loops on which a function is permissible.

Definition 7.10. We say that a function $\psi \in R(D)$ with constant slopes along bridges is a *new permissible function* on the first loop on which ψ is permissible. If γ_k is not the first loop in a block, then ψ is new if and only if it is permissible and $s_k(\psi) < s_k(\theta)$.

We say that ψ is a *departing permissible function* on γ_k if it is permissible and $s_{k+1}(\psi) > s_k(\theta)$. If γ_k is not the last loop in a block, then ψ is departing if and only if γ_k is the last loop on which ψ is permissible.

The main argument in the proof of Theorem 7.1 is an algorithm for constructing an independence among permissible functions on each block. We prepare for this with the following lemmas controlling the new and departing permissible functions on loops within a given block.

Lemma 7.11. *If γ_k is not the first loop of a block, then there is at most one new permissible function φ_{ij} on γ_k. Furthermore, if γ_k is lingering then there are none.*

Note that the first loops of the blocks are γ_1, γ_{z+1}, and $\gamma_{z'+1}$, so the conclusion of the lemma holds for $k \notin \{1, z+1, z'+1\}$.

Proof. Recall that, by Lemma 7.7, φ_{ij} is permissible on γ_k if and only if $s_k(\varphi_{ij}) \leq s_k(\theta) \leq s_{k+1}(\varphi_{ij})$. Suppose γ_k is not the first loop of a block. If φ_{ij} is a new permissible function, we must have $s_k(\varphi_{ij}) < s_k(\theta) \leq s_{k+1}(\varphi_{ij})$. Hence the outgoing slope of φ_i or φ_j must be strictly greater than the incoming slope. If γ_k is lingering then there is no such function, and hence there is no new permissible function.

Suppose γ_k is nonlingering. Then there is exactly one index i such that $s_{k+1}(\varphi_i) > s_k(\varphi_i)$, and the increase in slope is exactly 1. Note that the slopes of all other φ_j are unchanged, and different from both $s_k(\varphi_i)$ and $s_{k+1}(\varphi_i)$. Hence there is at most one j (possibly equal to i) such that $s_k(\varphi_{ij}) < s_k(\theta)$ and $s_{k+1}(\varphi_{ij}) \geq s_k(\theta)$, and at most one new permissible function φ_{ij}. □

Lemma 7.12. *There is at most one departing permissible function φ_{ij} on each loop γ_k. Furthermore, if γ_k is lingering then there are none.*

Proof. Suppose φ_{ij} is a departing permissible function on γ_k. Then $s_k(\varphi_{ij}) \leq s_k(\theta) < s_{k+1}(\varphi_{ij})$. Hence the slope of φ_i or φ_j must increase from β_k to β_{k+1}. If γ_k is lingering then there is no such function, and hence there is no departing permissible function. The rest of the proof is similar to that of the previous lemma. □

An important feature of our algorithm is that, when we reach the end of the block where θ has slope s, there is exactly one remaining unassigned function with slope s along the next bridge, and it can be assigned to achieve the minimum on the first half of that bridge. In order to verify this property, we must keep track of the non-lingering loops where there are no new permissible functions. These will be the loops numbered $b(T)$ and $b'(T)$, characterized as follows.

Definition 7.13. Let $b(T)$ be the 7th smallest entry appearing in the first two rows of the tableau T and let $b'(T)$ be the 8th smallest symbol appearing in the union of the first and third row.

When T is fixed, we omit it from the notation, and write simply b and b'. We note that

$$z < b < b' \leq z'.$$

The first two inequalities are straightforward. To see the last inequality, recall from Definition 7.3 that $z'+2$ is the 10th smallest entry that appears in the union of the second and third row. Therefore, the 9th smallest symbol appearing in the union of the second and third row must be strictly between b' and $z'+2$. From these inequalities, it follows that γ_b and $\gamma_{b'}$ are in the second block.

Example 7.14. In Example 7.2, we have $b(T) = 9$ and $b'(T) = 11$. Note that γ_9 and γ_{11} are the loops in the second block where the number of unassigned permissible functions dropped from 4 to 3 and from 3 to 2, respectively.

Lemma 7.15. *If $b \neq z+1$, then the non-lingering loops with no new permissible functions are exactly γ_z, γ_b, $\gamma_{b'}$, and $\gamma_{z'+2}$. Otherwise, the non-lingering loops with no new permissible functions are exactly γ_z, $\gamma_{b'}$, and $\gamma_{z'+2}$, and there are only 3 permissible functions on γ_b.*

Proof. We begin by showing that there are no new permissible functions on γ_z. Suppose φ_{ij} is a new permissible function on γ_z, i.e., $s_z(\varphi_{ij}) < 4 \leq s_{z+1}(\varphi_{ij})$. We show that this is impossible.

Recall that z is the 6th smallest entry appearing in the first two rows of T (Definition 7.3). There are 4 possibilities for the location of these entries, corresponding to partitions of 6 into no more than 2 parts. We consider the case where the partition is $(4, 2)$; the other three cases are similar.

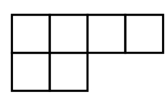

FIGURE 11. The Young diagram corresponding to the partition $(4, 2)$.

When the 6 smallest entries in the first two rows occupy the Young diagram corresponding to $(4, 2)$, then z is either the 4th entry in the first row, or the 2nd entry in the second row. Suppose z is the 2nd entry in the second row. Then $s_{z+1}(\Sigma) = s_z(\Sigma) + (0, 0, 0, 0, 0, 1, 0)$, and $s_{z+1}(\Sigma)$ is either

$$(-3, -2, -1, 1, 2, 4, 5) \text{ or } (-3, -2, -1, 1, 2, 4, 6).$$

By inspection, there is no pair of indices (i, j) such that $s_z[i] + s_z[j] < 4$ and $s_{z+1}[i] + s_{z+1}[j] \geq 4$, so there is no new permissible function on γ_z. Similarly, if z is the 4th entry in the first row, then $s_{z+1}(\Sigma) = s_z + (0, 0, 0, 1, 0, 0, 0)$, and $s_{z+1}(\Sigma)$ is one of the following:

$$(-3, -2, -1, 1, 2, 4, 5), \ (-3, -2, -1, 1, 2, 4, 6), \ \text{ or } \ (-3, -2, -1, 1, 2, 5, 6).$$

Once again, by inspection, there is no new permissible function on γ_z.

The proofs that $\gamma_{b'}$ and $\gamma_{z'}$ have no new permissible functions are similar, as is the proof that γ_b has no new permissible functions if b is not the first loop in the second block, i.e., if $b \neq z+1$. If $b = z+1$, then a similar argument shows that there is no permissible function φ_{ij} with $s_b(\varphi_{ij}) < 3$, and a case-by-case examination shows that there are only 3 permissible functions on γ_b.

It remains to prove that these are the only loops with no new permissible functions. We do so by a counting argument. For simplicity, suppose the first loop of each block is not lingering and

$b \neq z+1$. We claim that there are exactly 3 permissible functions on γ_1, 4 permissible functions on γ_{z+1}, and 3 permissible functions on $\gamma_{z'+1}$. On γ_1, the permissible functions are φ_{36}, φ_{46}, and φ_{55}. The precise permissible functions on γ_{z+1} and $\gamma_{z'+1}$ depend on the different possibilities for $s_{z+1}(\Sigma)$, as discussed above, and the analogous possibilities for $s_{z'+1}(\Sigma)$, but the claim can be verified case-by-case

There are 4 functions that are not permissible on any loop, in addition to the 10 functions that are permissible on the first loop of some block. Each of the remaining 14 functions must be new on some other non-lingering loop. By Lemma 7.11 there is at most one on each of the non-lingering loops other than γ_z, γ_b, $\gamma_{b'}$, and $\gamma_{z'+2}$, which have none. There are precisely 14 such loops. Hence each has a new permissible function, as required. The remaining cases where the first loops of some blocks may be lingering and where b may be equal to $z+1$ are similar, with only minor modifications to the details (e.g., if γ_1 is lingering, then there are exactly 2 new permissible functions on γ_1). □

Corollary 7.16. *On each of the three blocks, the number of permissible functions is 1 more than the number of non-lingering loops.*

Remark 7.17. It may be more natural to define the blocks by setting z to be the smallest positive integer so that the number of permissible functions on the 1st block is 1 more than the number of loops, and similarly for z'. A simple counting argument shows that such values z, z' exist. Lemma 7.18 below is the only piece of this argument that requires more precise information about the values z and z'.

Another important feature of our algorithm is that, when we arrive at a loop γ_k with no departing permissible functions, if we choose coefficients so that the unassigned permissible functions achieve the minimum at the righthand endpoint, then one of them achieves the minimum uniquely on a segment of the upper half of the loop. Our verification of this property depends on knowing that there are at most three non-departing permissible functions.

Lemma 7.18. *For any loop γ_k, there are at most 3 non-departing permissible functions on γ_k.*

Proof. If φ_{ij} is a non-departing permissible function on γ_k, then $s_{k+1}(\varphi_{ij}) = s_k(\theta)$. For each i, this equality holds for at most one j. It follows that there are most $\lceil \frac{r+1}{2} \rceil$ non-departing permissible functions φ_{ij}. Furthermore, equality is possible only when $2s_{k+1}(\varphi_3) = s_k(\theta)$. We claim that this never happens. Indeed, if $k \leq z$, then $2s_{k+1}(\varphi_3) \leq 2 < s_k(\theta)$. If $z < k \leq z'$, then $s_k(\theta)$ is odd. And if $k > z'$ then $2s_{k+1}(\varphi_3) \geq 4 > s_k(\theta)$. This proves the claim. □

Remark 7.19. Lemma 7.18 and its generalization Lemma 8.23 are the key places where we use the assumption that $r = 6$. Extending our method to prove further cases of the Strong Maximal Rank Conjecture for larger r would require new ideas at these steps.

Proposition 7.20. *Consider a set of at most three non-departing permissible functions from the set $\{\varphi_{ij}\}$ on a loop γ_k and assume that all of the functions take the same value at w_k. Then there is a point of γ_k at which one of these functions is strictly less than the others.*

Our proof of this proposition relies on the following fact about the divisor of a piecewise linear function on Γ obtained as the minimum of several functions in $R(D)$.

Shape Lemma for Minima. [JP14, Lemma 3.4] *Let D be a divisor on a metric graph Γ, with $\psi_0, \ldots, \psi_r$ piecewise linear functions in $R(D)$, and let*

$$\theta = \min\{\psi_0, \ldots, \psi_r\}.$$

Let $\Gamma_j \subseteq \Gamma$ be the closed set where θ is equal to ψ_j. Then $\operatorname{div}(\theta) + D$ contains a point $v \in \Gamma_j$ if and only if v is in either

(1) *the support of the divisor $\operatorname{div}(\psi_j) + D$, or*
(2) *the boundary of Γ_j.*

Proof of Proposition 7.20. We will consider the case where there are exactly three functions φ, φ' and φ''. The cases where there are one or two functions follows from a similar, but simpler, argument. Since φ is permissible on γ_k, we have $s_k(\varphi) \leq s_k(\theta)$, and since φ is non-departing, we have $s_{k+1}(\varphi) = s_k(\theta)$. The same holds for φ' and φ''.

Let ϑ be the pointwise minimum of the three functions. Since the slope of ϑ along any tangent direction agrees with that of one of the three, the incoming slope from the left at v_k is at most the outgoing slope to the right at w_k, which is equal to $s_k(\theta)$.

It follows that the restriction $(D + \mathrm{div}(\vartheta))_{|\gamma_k}$ has degree at most 2. Hence $\gamma_k \smallsetminus \mathrm{Supp}(D + \mathrm{div}(\vartheta))$ consists of at most two connected components. By the Shape Lemma for Minima, the boundary points of a region where a function φ achieves the minimum are contained in the support. Therefore, the region where any one of the functions φ achieves the minimum is either one of these connected components, or the union of both.

Since all three functions agree at w_k, and no two functions agree on the whole loop γ_k, we can narrow down the combinatorial possibilities as follows: either all three functions agree on one region that contains w_k and one of the three achieves the minimum uniquely on the other region, or two different pairs of functions agree on the two different regions, and w_k is in the boundary of both. These two possibilities are illustrated in Figure 12.

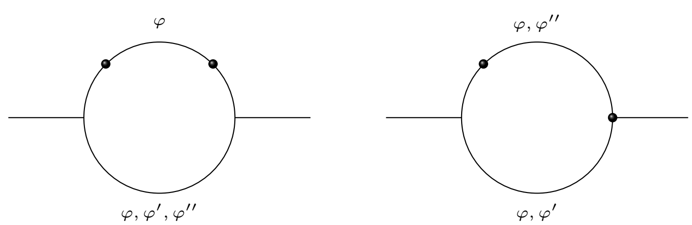

FIGURE 12. Two possibilities for where 3 functions may achieve the minimum.

We now rule out the possibility illustrated on the right in Figure 12, where w_k is in the boundary of both regions. Note that one function, which we may assume to be φ, achieves the minimum on all of γ_k. Furthermore, all three functions have the same slope along the outgoing bridge β_{k+1}, so φ also achieves the minimum on the bridge. Therefore ϑ is equal to φ in a neighborhood of w_k. However, $D + \mathrm{div}(\vartheta)$ contains w_k and $D + \mathrm{div}(\varphi)$ does not, a contradiction.

We conclude that the minimum is achieved as depicted on the left in Figure 12, with all three functions achieving the minimum on a region that includes w_k, and one function achieving the minimum uniquely on the other region. This proves the lemma. $\square$

7.5. **Algorithm for constructing an independence.** We now prove Theorem 7.1, by giving an algorithm for constructing an independence $\theta = \min_{ij}\{\varphi_{ij} + c_{ij}\}$ in the vertex avoiding case, with slopes as specified in §7.2. In this algorithm, we move from left to right across each of the three blocks where $s_k(\theta)$ is constant, adjusting the coefficients of unassigned permissible functions and assigning one function φ_{ij} to each non-lingering loop so that each function achieves the minimum on the loop to which it is assigned. At the end of each block, there is one remaining unassigned permissible function that achieves the minimum on the bridge between blocks, which we assign to that bridge. Since there are 21 nonlingering loops and three blocks, this gives us an independent configuration of 24 functions. The remaining 4 functions, with slopes too high or too low to be permissible on any block, achieve the minimum uniquely on the bridges to the left of the first loop or to the right of the last loop, respectively. Example 7.2 illustrates the procedure for one randomly chosen tableau. We now list a few of the key properties of the algorithm:

(i) Once a function has been assigned to a bridge or loop, it always achieves the minimum uniquely at some point on that bridge or loop (Lemma 7.22).

(ii) A function never achieves the minimum on any loop to the right of the bridge or loop to which it is assigned (Lemma 7.21).
(iii) The coefficient of each function is initialized to ∞ and then assigned a finite value when the function is assigned to a bridge or becomes permissible on a loop, whichever comes first.
(iv) After the initial assignment of a finite coefficient, subsequent adjustments to this coefficient are smaller and smaller perturbations. This is related to the fact that the edges get shorter and shorter as we move from left to right across the graph.
(v) Only the coefficients of unassigned functions are adjusted, and all adjustments are upward. This ensures that once a function is assigned and achieves the minimum uniquely on a loop, it always achieves the minimum uniquely on that loop.
(vi) Exactly one function is assigned to each of the 21 non-lingering loops, and the remaining seven functions are assigned to either the leftmost bridge, the rightmost bridge, or one of the two bridges between the blocks.

The algorithm terminates when we reach the rightmost bridge, at which point each of the 28 functions $\varphi_{ij} + c_{ij}$ achieves the minimum uniquely at some point on the graph.

Start at the first bridge. Start at β_1 and initialize $c_{66} = 0$. Initialize c_{56} so that $\varphi_{56} + c_{56}$ equals φ_{66} at a point one third of the way from w_0 to v_1. Initialize c_{55} and c_{46} so that $\varphi_{55} + c_{55}$ and $\varphi_{46} + c_{46}$ agree with $\varphi_{56} + c_{56}$ at a point two thirds of the way from w_0 to v_1. Initialize all other coefficients c_{ij} to ∞. Note that φ_{66} and φ_{56} achieve the minimum uniquely on the first and second third of β_1, respectively. Assign both of these functions to β_1, and proceed to the first loop.

Loop subroutine. Each time we arrive at a loop γ_k, apply the following steps.

Loop subroutine, step 1: Re-initialize unassigned coefficients. Suppose γ_k is non-lingering. Note that there are at least two unassigned permissible functions, by Lemma 7.15. Find the unassigned permissible function φ_{ij} that maximizes $\varphi_{ij}(w_k) + c_{ij}$. Initialize the coefficients of the new permissible functions (if any) and adjust the coefficients of the other unassigned permissible functions upward so that they all agree with φ_{ij} at w_k. (The unassigned permissible functions are strictly less than all other functions on γ_k, even after this upward adjustment; see Lemma 7.21.)

Loop subroutine, step 2: Assign departing functions. If there is a departing function, assign it to the loop. (There is at most one, by Lemma 7.12.) Adjust the coefficients of the other permissible functions upward so that all of the functions agree at a point on the following bridge a short distance to the right of w_k, but far enough so that the departing function achieves the minimum uniquely on the whole loop. This is possible because the bridge is much longer than the edges in the loop. Proceed to the next loop.

Loop subroutine, step 3: Skip lingering loops. If γ_k is a lingering loop, do nothing and proceed to the next loop.

Loop subroutine, step 4: Otherwise, use Proposition 7.20. By Lemma 7.18, there are at most 3 non-departing functions. By Proposition 7.20, there is one φ_{ij} that achieves the minimum uniquely at some point of γ_k. We adjust the coefficient of φ_{ij} upward by $\frac{1}{3}m_k$. This ensures that it will never achieve the minimum on any loops to the right, yet still achieves the minimum uniquely on this loop; see Lemma 7.21, below. Assign φ_{ij} to γ_k, and proceed to the next loop.

Proceeding to the next loop. If the next loop is contained in the same block, then move right to the next loop and apply the loop subroutine. Otherwise, the current loop is the last loop in its block. In this case, proceed to the next block.

Proceeding to the next block. After applying the loop subroutine to the last loop in a block, there is exactly one unassigned permissible function. (This follows from Corollary 7.16.) The unassigned permissible function φ_{ij} achieves the minimum uniquely on the beginning of the outgoing bridge, without any further adjustment of coefficients. Assign φ_{ij} to this bridge.

If we are at the last loop γ_g, then proceed to the last bridge. Otherwise, there are several new permissible functions on the first loop of the next block, as detailed in Lemma 7.18, above. Initialize the coefficient of each new permissible function so that it is equal to θ at the midpoint of the bridge between the blocks, and then apply the loop subroutine.

The last bridge. Initialize the coefficient c_{01} so that $\varphi_{01}+c_{01}$ equals θ at the midpoint of the last bridge β_{g+1}. Initialize c_{00} so that $\varphi_{00}+c_{00}$ equals θ halfway between the midpoint and the rightmost endpoint. Note that both of these functions now achieve the minimum uniquely at some point on the second half of β_{g+1}. Assign both of these functions to β_{g+1}, and output $\theta = \min_{ij}\{\varphi_{ij} + c_{ij}\}$.

7.6. **Verifying the algorithm.** We now prove that the output $\theta = \min_{ij}\{\varphi_{ij}+c_{ij}\}$ is an independence.

Lemma 7.21. *Suppose that φ_{ij} is assigned to the loop γ_k or the bridge β_k. Then φ_{ij} does not achieve the minimum at any point to the right of v_{k+1}.*

Proof. If φ_{ij} is assigned to the bridge β_k, then γ_k is the last loop in a block. It follows that φ_{ij} is not permissible on any loop to the right of γ_k. It therefore cannot achieve the minimum anywhere to the right of β_k. Similarly, if φ_{ij} is assigned to γ_k and departing on γ_k, then it is not permissible on loops $\gamma_{k'}$ for $k' > k$. Thus, it cannot achieve the minimum on any of the loops to the right of γ_k.

Otherwise, if φ_{ij} is assigned to γ_k and not departing on γ_k, then since φ_{ij} is permissible, the difference $\varphi_{ij}(v) - \theta(v)$ is at least

$$\frac{1}{3}m_k - d\sum_{t=k+1}^{k'} m_t,$$

for any point $v \in \gamma_{k'}$ with $k' > k$. By our assumptions on edge lengths, this expression is positive. □

Lemma 7.22. *Suppose that φ_{ij} is assigned to the loop γ_k. Then there is a point $v \in \gamma_k$ where φ_{ij} achieves the minimum uniquely.*

Proof. If there is an unassigned departing function φ_{ij} on γ_k, then by construction this is the only function that achieves the minimum at w_k. Otherwise, any two permissible functions differ by an integer multiple of $\frac{1}{2}m_k$ at points whose distance from w_k is a half-integer multiple of m_k. By construction, there is such a point where the assigned function φ_{ij} achieves the minimum uniquely and, after increasing the coefficient by $\frac{1}{3}m_k$, it still does. □

Proof of Theorem 7.1. By construction, the functions φ_{66} and φ_{56} achieve the minimum on the bridge β_1, and the functions φ_{00} and φ_{01} achieve the minimum on the bridge β_{g+1}.

We first show that every non-lingering loop has an assigned function. To see this, suppose that there is a non-lingering loop with no assigned function, and let γ_k be the first such loop. Consider the set of functions φ_{ij} that are permissible on the block containing γ_k. If φ_{ij} is permissible on a loop $\gamma_{k'}$ in this block with $k' < k$, then there are two possibilities: either φ_{ij} is also permissible on γ_k, or it is not. In either case, we see that φ_{ij} must be assigned to a loop $\gamma_{k''}$ with $k'' < k$. Indeed, if φ_{ij} is permissible on γ_k, then it must be assigned to an earlier loop, otherwise the algorithm assigns a function to γ_k. If not, then there is a $k'' < k$ such that φ_{ij} is a departing permissible function on $\gamma_{k''}$. By construction, this function must be assigned to loop $\gamma_{k''}$, or an earlier loop. It follows that the number of such functions is at most the number of non-lingering loops $\gamma_{k'}$ with $k' < k$.

Now, for every other function φ_{ij} that is permissible on the block containing γ_k, there must exist a non-lingering loop $\gamma_{k'}$, with $k' > k$, such that φ_{ij} is a new permissible function on $\gamma_{k'}$. Since there is at most one new permissible function per loop, we see that the number of permissible functions on the block containing γ_k is fewer than the number of non-lingering loops. By Corollary 7.16, however, this is impossible.

Indeed, by Corollary 7.16, the number of permissible functions on each of the 3 blocks is exactly one more than the number of non-lingering loops in that block. Every non-lingering loop has an assigned function, and the remaining function achieves the minimum at the bridge immediately to the right of the block. The result then follows from Lemma 7.22. □

Remark 7.23. Note that the functions $\{\varphi_{ij}\}$ may admit many different independences with different combinatorial properties. There is no obvious reason to prefer one such combination over another. We present one particular algorithm for constructing an independence that works uniformly for all vertex avoiding divisors and generalizes naturally to the non-vertex avoiding case. A variant of the algorithm produces an independence where each of the functions φ_{ij} achieves the minimum uniquely somewhere to the right of the second loop. Specifically, for any vertex avoiding divisor class D, note that the functions assigned to β_1, γ_1 and γ_2 all have distinct slopes greater than 4 along the bridge β_3. We could therefore adjust the algorithm by first assigning all of these functions to β_3, initalizing the coefficients of the functions with slope 4 so that they agree at the midpoint of β_3, and then picking up the algorithm by proceeding to the loop γ_3. This variant will be important for the proof of Theorem 1.4 in §10.

8. Building blocks and the master template

We now begin the construction of a tropical independence from 28 pairwise sums of functions in Σ when the tropical divisor class $[D]$ is not vertex avoiding. The construction has two main steps. First we build a piecewise linear function θ from pairwise sums of function in $R(D)$ (but not necessarily in Σ), which we call the *master template*. Then we identify 28 pairwise sums of functions in Σ that form an independence when fit to this template, via best approximation from above.

8.1. **Overview of our proof in the non vertex avoiding case.** Let us explain the idea of this construction in advance. Fix $\theta \in \mathrm{PL}(\Gamma)$. Imagine trying to approximate θ from above by tropical linear combinations of $\{\psi_1, \ldots, \psi_s\} \subseteq \mathrm{PL}(\Gamma)$. We claim that there is a best possible such approximation. Indeed, for $1 \leq i \leq s$, the function $\psi_i - \theta$ is continuous and bounded, so it achieves its minimum b_i on Γ. Then $\psi_i - b_i \geq \theta$, with equality at some point $v \in \Gamma$, and

$$\vartheta = \min_i \{\psi_i - b_i\}$$

is the smallest tropical linear combination of $\{\psi_1, \ldots, \psi_s\}$ that is greater than or equal to θ.

Roughly speaking, the main purpose of this section is the construction of a function $\theta \in R(2D)$, to be used as a template in this way, for the construction of an independence ϑ as a best approximation of θ from above, in the following section. The template θ is itself a tropical linear combination of pairwise sums of simpler functions in $R(D)$, that we call *building blocks*. These building blocks mimic the behavior of the distinguished functions $\varphi_0, \ldots, \varphi_6$ in the vertex avoiding case, and yet are flexible enough to adapt to a wide range of possibilities for Σ; they are analogous to the functions ψ_0, ψ_1, and ψ^∞ in Example 6.8. In §9, we will carry through a case-by-case analysis, depending on the combinatorial properties of Σ. In each case, we identify 28 pairwise sums of functions in Σ, and prove that the best possible approximation of θ from above, by a tropical linear combination of these 28 functions, is an independence.

More precisely, the main content of this section is an algorithm that takes as input a collection $\mathcal{A}$ of building blocks, and a collection $\mathcal{B}$ of pairwise sums of building blocks, both satisfying some necessary technical conditions, and outputs the template θ. The case-by-case analysis in §9 involves identifying the collections $\mathcal{A}$ and $\mathcal{B}$ in each case and verifying that they satisfy the necessary technical conditions to run the algorithm that constructs the template θ. Then, in each case, we identify the 28 functions that yield the independence when fit to the template.

The algorithm for constructing the template θ from the building blocks is closely analogous to the algorithm presented in §7. The verification of this algorithm is technical, and involves several new combinatorial notions invented solely for this purpose. Readers who are interested in understanding every detail will find it useful to compare each step with the corresponding step in the vertex avoiding case; indeed, in the vertex avoiding case, the building blocks are exactly the distinguished functions

$\varphi_0, \ldots, \varphi_6$ and the output of the algorithm in this section is precisely the tropical independence θ constructed in §7. Presenting the algorithm in the vertex avoiding case separately was not logically necessary, but we have done so with the hope that readers may use this simpler special case for guidance in understanding our general constructions.

8.2. **Linear series with imposed ramification.** Recall that $r = 6$, $g = 21 + \rho$, and $d = 24 + \rho$. Throughout the rest of the paper, we assume that $\rho \leq 2$. For the proof of Theorem 1.3, completed in §9, it is enough to consider complete linear series on a smooth curve of genus g over K whose skeleton is a chain of g loops with admissible edge lengths. However, in the proof of Theorem 1.4, which will be completed in §10, we apply similar methods to study incomplete linear series of degree d and rank r on curves of genus $g' = g - 1$ or $g - 2$, whose skeleton is a chain of g' loops with admissible edge lengths, with appropriate ramification conditions at a point specializing to the left endpoint w_0. Since the arguments are similar in all of these cases, we treat them in parallel.

Let X be a curve of genus g' over our nonarchimedean field K whose skeleton Γ is a chain of g' loops with admissible edge lengths. Let D_X be a divisor of degree d on X, with a linear series $V \subseteq H^0(X, \mathcal{O}(D_X))$ of rank r, and let $\Sigma = \mathrm{trop}(V)$. We let $p \in X$ be a point specializing to w_0. In this section and in §9, we assume further that one of the following three conditions holds:

(i) $g' = g$,
(ii) $g' = g - 1$ and $a_1^V(p) \geq 2$, or
(iii) $g' = g - 2$ and either $a_1^V(p) \geq 3$, or $a_0^V(p) + a_2^V(p) \geq 5$.

We define the multiplicities of loops and bridges, and the ramification weights at the left and right endpoints w_0 and $v_{g'+1}$, just as in the case $g' = g$ (Definitions 6.16 and 6.17). Let

$$\rho' := \rho(g', r, d) = g' - (r + 1)(g' - d + r).$$

Our hypotheses on the ramification sequence of V at p give the following analogue of Proposition 6.18.

Lemma 8.1. *We have*

$$\mathrm{wt}(v_{g'+1}) + \sum_{k=1}^{g'} \mu(\gamma_k) + \sum_{k=1}^{g'+1} \mu(\beta_k) \leq \rho.$$

In particular, the sum of the multiplicities of the loops and bridges is at most $\rho \leq 2$.

Proof. By Proposition 6.18, we have

$$\mathrm{wt}(w_0) + \mathrm{wt}(v_{g'+1}) + \sum_{k=1}^{g'} \mu(\gamma_k) + \sum_{k=1}^{g'+1} \mu(\beta_k) \leq \rho',$$

and by (29), $\mathrm{wt}(w_0)$ is greater than or equal to the ramification weight of V at p. It therefore suffices to show that this ramification weight is greater than or equal to $\rho' - \rho$. If $g' = g$, then $\rho' = \rho$, and there is nothing to show.

If $g' = g - 1$, then $\rho' = \rho + 6$. Since $a_1^V(p) \geq 2$, we have $a_i^V(p) \geq i + 1$ for all $i \geq 1$, so the ramification weight of V at p is at least 6.

Finally, if $g' = g - 2$, then $\rho' = \rho + 12$. If $a_1^V(p) \geq 3$, then $a_i^V(p) \geq i + 2$ for all $i \geq 1$, so the ramification weight of V at p is at least 12. On the other hand, if $a_0^V(p) + a_2^V(p) \geq 5$, then since $a_2^V(p) \geq a_0^V(p) + 2$, we have $a_2^V(p) \geq 4$. If equality holds, then $a_0^V(p) \geq 1$, $a_1^V(p) \geq 2$, and $a_i^V(p) \geq i + 2$ for all $i \geq 2$, so the ramification weight of V at p is at least 12. Finally, if $a_2^V(p) > 4$, then $a_i^V(p) \geq i + 3$ for all $i \geq 2$, so the ramification weight of V at p is at least $15 > 12$. □

8.3. **Building Blocks.** Roughly speaking, the building blocks are the functions in $R(D)$ that have constant slope along each bridge and behave as much as possible like the functions φ_i in the vertex avoiding case, while respecting the constraints on functions in Σ imposed by the slope vectors. The simplest are those functions φ whose incoming and outgoing slopes at each loop γ_k satisfy $s_k[\varphi] = s_k[i]$ and $s'_k[\varphi] = s_k[i]$, for some fixed i. Our definition is motivated by the extremals of [HMY12] and by Example 6.8.

Before giving the general definition, we introduce the useful auxiliary notion of *incoming and outgoing slope indices*[2] These indices account for how the incoming and outgoing slopes of a function $\varphi \in R(D)$ relate to the slope vectors s_k and s'_k determined by Σ, adjusted for any bends at v_k and w_k. Let $d_{v_k}(\varphi) = \deg_{v_k}(D + \mathrm{div}(\varphi))$ and $d_{w_k}(\varphi) = \deg_{w_k}(D + \mathrm{div}(\varphi))$. If $0 < k < g' + 1$, then the *incoming slope index* $\tau_k(\varphi)$ is

$$\tau_k(\varphi) := \min\{i \,:\, s_k[i] \geq s_k(\varphi) - d_{v_k}(\varphi)\}.$$

Similarly, the *outgoing slope index* $\tau'_k(\varphi)$ is

$$\tau'_k(\varphi) := \max\{i \,:\, s'_k[i] \leq s'_k(\varphi) + d_{w_k}(\varphi)\}.$$

We also define

$$\tau'_0(\varphi) := \max\{i \,:\, s'_0[i] \leq s'_0(\varphi)\},$$
$$\tau_{g'+1}(\varphi) := \min\{i \,:\, s_{g'+1}[i] \geq s_{g'+1}(\varphi)\}.$$

Intuitively, one may think that a function with incoming (resp. outgoing) slope index i behaves most like a typical function in Σ with incoming slope $s_k[i]$ (resp. a typical function in Σ with outgoing slope $s'_k[i]$), near the left hand side (resp. right hand side) of the loop γ_k.

Remark 8.2. The integer $s_k(\varphi) - d_{v_k}(\varphi)$ is equal to the sum of the slopes of φ along the two rightward pointing tangent vectors based at v_k. Because of this, if two functions φ and ϕ differ by a constant on γ_k, then by definition we have $\tau_k(\varphi) = \tau_k(\phi)$. Similarly, if the restriction of φ and ϕ to γ_k differ by a constant, then $\tau'_k(\varphi) = \tau'_k(\phi)$.

Functions whose slope indices decrease when moving from left to right across the graph can be expressed as tropical linear combinations of functions with constant slopes along bridges whose slope indices do not decrease. Moreover, since we restrict our attention to the case $\rho \leq 2$, the classification of switching loops and bridges discussed in §6 ensures that the slope indices of functions in Σ never increase by more than 1 when crossing any loop or bridge, and only when that loop or bridge switches the relevant slope. For this reason, we only consider building blocks whose slope indices satisfy this condition. Finally, all functions $\varphi \in \Sigma$ satisfy the inequalities $s_k[\tau_k(\varphi)] \leq s_k(\varphi) \leq s'_{k-1}[\tau'_{k-1}(\varphi)]$ for all k, and it is technically convenient to impose this condition on building blocks as well.

Definition 8.3. A building block is a function $\varphi \in R(D)$ with constant slope along each bridge, whose slope index sequence $\tau'_0(\varphi), \tau_1(\varphi), \tau'_1(\varphi), \ldots, \tau'_{g'}(\varphi), \tau_{g'+1}(\varphi)$ is well-defined, nondecreasing, and satisfies

(i) $\tau'_k(\varphi) \leq \tau_k(\varphi) + 1$, with $\tau'_k(\varphi) = \tau_k(\varphi)$ if γ_k does not switch slope $\tau_k(\varphi)$;
(ii) $\tau_k(\varphi) \leq \tau'_{k-1}(\varphi) + 1$, with $\tau_k(\varphi) = \tau'_{k-1}(\varphi)$ if β_k does not switch slope $\tau'_{k-1}(\varphi)$;
(iii) $s_k[\tau_k(\varphi)] \leq s_k(\varphi) \leq s'_{k-1}[\tau'_{k-1}(\varphi)]$ for all k.

Note that this definition is sufficiently flexible so that, if β_k is a decreasing bridge, where $s'_{k-1}[i] > s_k[i]$, there are building blocks φ with constant slope index $\tau_k(\varphi) = \tau'_k(\varphi) = i$ for all k that have slope between $s'_{k-1}[i]$ and $s_k[i]$ on β_k. It is also sufficiently rigid so that, in the vertex avoiding case, the building blocks are precisely the distinguished functions φ_i, which have $\tau_k(\varphi_i) = \tau'_k(\varphi_i) = i$ for all k. In general, for any $0 \leq i \leq r$, there is a (not necessarily unique) building block ϕ with

[2]Strictly speaking, these slope indices are not defined for all functions in $R(D)$, since some functions in $R(D)$ may have higher or lower slopes on some bridges than any function in Σ. Nevertheless, they are defined for all functions in Σ, and for all of the functions that appear in our constructions.

constant slope index sequence $\tau_k(\phi) = \tau'_k(\phi) = i$. If there are no switching loops or bridges, then the slope index sequence of any building block is constant.

We find it helpful to have the following terminology, for specifying loops and bridges where the slope vectors decrease.

Definition 8.4. We say that γ_k is a *decreasing loop* if there is a value h such that $s_k[h] > s'_k[h]$. Similarly, we say that β_k is a *decreasing bridge* if there is a value h such that $s'_{k-1}[h] > s_k[h]$.

Note that all bridges with positive multiplicity are decreasing bridges, but not all loops with positive multiplicity are decreasing loops.

Remark 8.5. If φ is a building block and $s_k[\tau_k(\varphi)] \leq s'_k[\tau'_k(\varphi)]$ for all k, then $D + \mathrm{div}(\varphi)$ does not contain a smooth cut set, and it follows that $\varphi \in R(D)$ is an *extremal*, as defined in [HMY12]. On the other hand, if $s_k[\tau_k(\varphi)] > s'_k[\tau'_k(\varphi)]$ for some k, then φ is not necessarily an extremal. In such cases, γ_k must be a decreasing loop. In any case, every function in $R(D)$ can be written as a tropical linear combination of extremals, and hence our main constructions could be rephrased in terms of extremals. We find it simpler to work with building blocks, as defined above, since they are more closely tailored to the properties of Σ.

Example 8.6. The essential properties of building blocks are visible already when one considers tropicalizations of pencils of degree 2 on $\mathbf{P}^1$, as in Case 2 of Example 6.8. One may think of the segment Γ in this example as a chain of one loop, whose edges have length zero, located at the point x. It is a switching loop, and the possibilities for the slope index sequence of a building block are:

$$(\tau'_0(\varphi), \tau_1(\varphi), \tau'_1(\varphi), \tau_2(\varphi)) \in \{(0,0,0,0), (0,0,1,1), (1,1,1,1)\}.$$

There is a unique building block with each of these slope index sequences. In the example, these functions are denoted ψ_0, ψ^∞, and ψ_1, respectively (see Remark 6.9). Since Σ is the tropicalization of a pencil and these three functions are tropically independent, it is impossible for all of them to be in Σ. Nevertheless, every function in Σ can be written as a tropical linear combination of these three building blocks.

Example 8.7. Suppose that φ is a building block with $s'_k(\varphi) = s_k(\varphi) - 1$, i.e., the outgoing slope at γ_k is one less than the incoming slope. This is equivalent to

$$\deg(D + \mathrm{div}(\varphi))_{|\gamma_k} = 2.$$

There are three possible ways that this could happen, illustrated in Figure 13:

(i) β_k is a decreasing bridge, and $D + \mathrm{div}(\varphi)$ contains v_k;
(ii) γ_k is a decreasing loop;
(iii) β_{k+1} is a decreasing bridge, and $D + \mathrm{div}(\varphi)$ contains w_k.

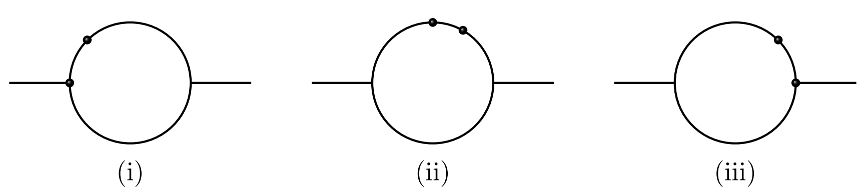

FIGURE 13. Three possibilities for $(D + \mathrm{div}(\varphi))_{|\gamma_k}$, when $s'_k(\varphi) = s_k(\varphi) - 1$

In the first and third cases, $\varphi_{|\gamma_k}$ is determined up to an additive constant. However, if γ_k is a decreasing loop, then there may be infinitely many possibilities for $(D+\mathrm{div}(\varphi))_{|\gamma_k}$, as φ ranges over building blocks with these slopes.

In our constructions, we will always start from a *finite* set of building blocks, chosen so that no two have both the same slope index sequences and also the same slopes along bridges.

8.4. **Sequences of partitions and slopes of the master template.** As in the vertex avoiding case, we construct the master template θ so that it has nearly constant slope on every bridge, with perhaps slope 1 or 2 higher for a short distance at the beginning of the bridge, and specify the nearly constant slopes $s_k(\theta)$ in advance. These slopes are most easily described in terms of partitions associated to the slope vectors $s_k(\Sigma) = (s_k[0], \ldots, s_k[r])$, as follows.

We associate to Σ a sequence of partitions (with possibly negative parts)

$$\lambda'_0, \lambda_1, \lambda'_1 \ldots, \lambda'_{g'}, \lambda_{g'+1},$$

each with at most $r+1$ columns, numbered from 0 to r. The $(r-i)$th column of λ_k contains $(g-d+r)+s_k[i]-i$ boxes. (Note that this convention depends on g, not g'.) Similarly, the $(r-i)$th column of λ'_k contains $(g-d+r)+s'_k[i]-i$ boxes. By Proposition 6.15, λ_k is a subset of λ'_{k-1}, and λ'_k contains at most one box that is not contained in λ_k. Moreover, $\lambda_{g'+1}$ contains the $(r+1)\times(g-d+r)$ rectangle. In the vertex avoiding case, each partition contains the partition that precedes it, and the sequence corresponds to the associated tableau discussed in §7.

As in the vertex avoiding case, we find it useful to specify in advance four indices z, z', b, and b' (cf. Definitions 7.3 and 7.13). These indices depend only on the sequence of partitions.

Definition 8.8. Let z be the largest integer such that λ'_z contains exactly 6 boxes in the union of the first two rows, and λ_z does not. Similarly, let z' be the largest integer such that $\lambda'_{z'+2}$ contains exactly 10 boxes in the union of the second and third row, and $\lambda_{z'+2}$ does not.

Let b be the largest integer such that λ'_b contains exactly 7 boxes in its first two rows, and λ_b does not. Similarly, let b' be the largest integer such that $\lambda'_{b'}$ contains exactly 8 boxes in the union of its first and third row, and $\lambda_{b'}$ does not.

Since each partition in the sequence contains at most 1 box not contained in the previous partition, such indices exist. In the vertex avoiding case, this definition agrees with Definitions 7.3 and 7.13.

Remark 8.9. The choice of z and z' in the vertex avoiding case is motivated by counting permissible functions; it ensures that there is precisely one unassigned permissible function at the end of each block, which is assigned to the following bridge. In the general case, the choice of z and z' is motivated by an analogous count of certain equivalence classes of permissible functions, which we call *cohorts* (see Definition 8.33). Our choice ensures that there is at most one unassigned cohort at the end of each block, which is assigned to the following bridge in the proof of Theorem 8.21.

As in §7.2, the slopes of the master template θ are given in terms of z and z' by:

$$s_k(\theta) = \begin{cases} 4 & \text{if } k \leq z \\ 3 & \text{if } z < k \leq z' \\ 2 & \text{if } z' < k \leq g'. \end{cases}$$

8.5. **Agreement on loops.** One essential difference between our constructions in this section and those in §7 is that here we may assign more than one function to a given loop, as long as all of the functions that we assign agree on that loop, in the following sense.

Definition 8.10. We say that functions *agree* on a subgraph of Γ if their restrictions to that subgraph differ by an additive constant.

We most often consider agreement on one loop at a time, but in a few key places, such as Definition 8.20, we also consider agreement on larger subgraphs. Agreement of functions on a given subgraph is an equivalence relation, and when the subgraph is clear from context we will refer to *equivalence classes* of functions that agree on that subgraph. We make some preliminary observations about sufficient conditions for two functions to agree on a loop, starting with the following analogue of Lemmas 7.11 and 7.12. New and permissible functions are defined as in Definition 7.5.

Lemma 8.11. *If γ_k is not the first loop in a block, then any two new permissible functions agree on γ_k. Similarly, any two departing permissible functions agree on γ_k.*

Proof. Recall that permissible functions have constant slope on bridges. If $\psi \in R(2D)$ is a new permissible function and γ_k is not the first loop in a block, then $s_{k+1}(\psi) \geq s_k(\theta) > s_k(\psi)$. If we write $s = s_k(\theta)$, then the restriction $\psi_{|\gamma_k}$ is in the linear series $R(E)$, where

$$E = 2D_{|\gamma_k} + (s-1)v_k - sw_k.$$

The divisor E has degree 1. On a loop, every divisor of degree 1 is equivalent to a unique effective divisor, so this determines $\psi_{|\gamma_k}$ up to an additive constant.

If ψ is departing, then $s_{k+1}(\psi) > s_k(\theta) \geq s_k(\psi)$, and the rest of the proof is similar. □

Corollary 8.12. *Let φ and φ' be building blocks. Assume that $\varphi + \varphi'$ is permissible on γ_k, which is not the first loop in a block, and that $s_k(\varphi) > s_k[\tau_k(\varphi)]$. Then there are new permissible functions on γ_k, and they all agree with $\varphi + \varphi'$.*

Proof. Since $s_k(\varphi)$ is strictly greater than $s_k[\tau_k(\varphi)]$, the degree of $D+\mathrm{div}(\varphi)$ at v_k is strictly positive. Consider a function $\phi \in \mathrm{PL}(\Gamma)$ that has the same slope index sequence as φ, and the same restriction to every loop, but with

$$s_k(\phi) = s_k(\varphi) - 1.$$

By construction, $\deg_{v_k}(D + \mathrm{div}(\phi)) = \deg_{v_k}(D + \mathrm{div}(\varphi)) - 1 \geq 0$. It follows that $\phi \in R(D)$ is a building block. Since

$$s_k(\phi + \varphi') < s_k(\varphi + \varphi') \leq s_k(\theta),$$

we see that $\phi + \varphi'$ is new on γ_k. It follows that $\varphi + \varphi'$ agrees with a new permissible function on γ_k. By Lemma 8.11, $\varphi + \varphi'$ agrees with every new permissible function on γ_k. □

With this in mind, we make the following definition.

Definition 8.13. We say that a sum of two building blocks $\varphi + \varphi'$ is *shiny* on γ_k if it is permissible on γ_k, and

$$s_k[\tau_k(\varphi)] + s_k[\tau_k(\varphi')] < s_k(\theta).$$

By Corollary 8.12, if γ_k is not the first loop in a block, then any new function on γ_k is shiny, as the terminology suggests. However, a shiny function is not necessarily new. Moreover, if γ_k is the first loop in a block, then there may be new functions that do not agree on γ_k, but all functions that are shiny on γ_k agree with each other. The following proposition describes their structure.

Proposition 8.14. *If $\psi = \varphi + \varphi'$ is shiny on γ_k, then the restriction of either $D + \mathrm{div}(\varphi)$ or $D + \mathrm{div}(\varphi')$ to $\gamma_k \smallsetminus \{v_k\}$ has degree 0. Moreover, either*

$$s_{k+1}(\varphi) > s_k[\tau_k(\varphi)] \text{ or } s_{k+1}(\varphi') > s_k[\tau_k(\varphi')].$$

Proof. If $s_k(\varphi) = s_k[\tau_k(\varphi)]$ and $s_k(\varphi') = s_k[\tau_k(\varphi')]$, then as in the proof of Lemma 8.11, we see that the restriction of $2D + \mathrm{div}(\psi)$ to γ_k has degree at most 1. Otherwise, by Corollary 8.12, ψ agrees with a new permissible function, and $2D + \mathrm{div}(\psi)$ contains v_k. Since ψ is permissible on γ_k, the restriction of $2D + \mathrm{div}(\psi)$ to γ_k has degree at most 2, hence the restriction of $2D + \mathrm{div}(\psi)$ to $\gamma_k \smallsetminus \{v_k\}$ has degree at most 1. It follows that the restriction of either $D + \mathrm{div}(\varphi)$ or $D + \mathrm{div}(\varphi')$ to $\gamma_k \smallsetminus \{v_k\}$ must have degree 0. Without loss of generality we may assume that the restriction of $D+\mathrm{div}(\varphi)$ to $\gamma_k \smallsetminus \{v_k\}$ has degree 0. Since $\deg_{w_k}(D+\mathrm{div}(\varphi)) = 0$, we see that $s_{k+1}(\varphi) = s'_k[\tau'_k(\varphi)]$.

If $D + \mathrm{div}(\varphi)$ does not contain v_k, then

$$s_{k+1}(\varphi) > s_k(\varphi) = s_k[\tau_k(\varphi)].$$

On the other hand, if $D + \mathrm{div}(\varphi)$ contains v_k, then

$$s_{k+1}(\varphi) = s_k(\varphi) > s_k[\tau_k(\varphi)].$$

In either case, we see that $s_{k+1}(\varphi) > s_k[\tau_k(\varphi)]$, as required. □

Remark 8.15. In the argument above, we assume that the restriction of $D + \mathrm{div}(\varphi)$ to $\gamma_k \smallsetminus \{v_k\}$ has degree 0, and show that $s_{k+1}(\varphi) > s_k[\tau_k(\varphi)]$. The converse is also true; if a building block φ satisfies $s_{k+1}(\varphi) > s_k[\tau_k(\varphi)]$, then the restriction of $D + \mathrm{div}(\varphi)$ to $\gamma_k \smallsetminus \{v_k\}$ has degree 0.

Lemma 8.16. *Let $\psi = \varphi + \varphi'$ and $\psi' = \phi + \phi'$ be shiny functions on γ_k. Then, after possibly reordering ϕ and ϕ', we have $\tau_k(\varphi) = \tau_k(\phi)$ and $\tau_k(\varphi') = \tau_k(\phi')$.*

Proof. By Proposition 8.14, we may assume after possibly reordering that the restrictions of both $D + \mathrm{div}(\varphi)$ and $D + \mathrm{div}(\phi)$ to $\gamma_k \smallsetminus \{v_k\}$ have degree zero. It follows that φ and ϕ agree on γ_k. By Lemma 8.11, $\varphi + \varphi'$ agrees with $\phi + \phi'$ on γ_k, hence φ' agrees with ϕ' on γ_k as well. The equality of slope indices follows from Remark 8.2. □

In §8.7 below, we state Theorem 8.21, which gives the essential properties of the master template, constructed as a tropical linear combination of a collection $\mathcal{B}$ of pairwise sums of building blocks. We should stress that the hypotheses of this theorem are as important as the conclusions; we need a few technical conditions on the collection $\mathcal{B}$ of pairwise sums of building blocks in order to successfully run the algorithm to construct the master template. In §9, we will consider several cases depending on the properties of Σ, and in each case we will choose a set $\mathcal{B}$ and show that it satisfies these properties.

We begin with a technical property on the collection of building blocks to be used as summands.

Definition 8.17. Let $\mathcal{A}$ be a set of building blocks. We say that $\mathcal{A}$ satisfies property (**A**) if any two functions $\varphi, \varphi' \in \mathcal{A}$ with $\tau'_k(\varphi) = \tau'_k(\varphi')$ agree on γ_k, and no two functions in $\mathcal{A}$ differ by a constant.

Note that there are only finitely many possibilities for the slope of a building block on each bridge. It follows that any collection of building blocks that satisfies (**A**) is necessarily finite.

Before stating the other technical properties, we note that this definition has the following important consequences.

Lemma 8.18. *Let $\mathcal{A}$ be a set of building blocks satisfying (**A**). Let $\varphi, \varphi' \in \mathcal{A}$, and suppose that $s_{k+1}(\varphi) = s_{k+1}(\varphi')$ and $\mu(\beta_{k+1}) \leq 1$. Then φ and φ' agree on γ_k.*

Proof. If φ and φ' disagree on γ_k, then $\tau'_k(\varphi) \neq \tau'_k(\varphi')$. Therefore, there must be $j \neq j'$ such that

$$s'_k[j] \geq s_{k+1}(\varphi) \geq s_{k+1}[j] \text{ and } s'_k[j'] \geq s_{k+1}(\varphi') \geq s_{k+1}[j'].$$

Assume without loss of generality that $j' > j$. Then

$$s'_k[j'] > s'_k[j] \geq s_{k+1}[j'] > s_{k+1}[j],$$

which implies that the bridge β_{k+1} has multiplicity at least 2. □

Lemma 8.19. *Let $\mathcal{A}$ be a set of building blocks satisfying (**A**), and let $\varphi, \varphi' \in \mathcal{A}$ satisfy*

$$\tau_k(\varphi) = \tau_k(\varphi') \text{ and } \tau_{k+1}(\varphi) = \tau_{k+1}(\varphi').$$

Then $\tau'_k(\varphi) = \tau'_k(\varphi')$ and φ agrees with φ' on γ_k.

Proof. Because a switching loop has multiplicity at least 1 and a switching bridge has multiplicity at least 2, we cannot have both that γ_k is a switching loop and β_{k+1} is a switching bridge. It follows that either

$$\tau'_k(\varphi) = \tau_k(\varphi) \text{ and } \tau'_k(\varphi') = \tau_k(\varphi')$$

or

$$\tau'_k(\varphi) = \tau_{k+1}(\varphi) \text{ and } \tau'_k(\varphi') = \tau_{k+1}(\varphi').$$

Thus, $\tau'_k(\varphi) = \tau'_k(\varphi')$. By property (**A**), it follows that φ agrees with φ' on γ_k. □

8.6. **Properties (B) and (B′).** We now introduce two key technical properties for collections of pairwise sums of building blocks. These conditions will be essential for our algorithmic construction of the tropical linear combination of pairwise sums of building blocks, which we call the master template, below. We write $\Gamma_{\leq k}$ for the subgraph of Γ to the left of w_k, i.e., $\Gamma_{\leq k}$ is the union of the loops γ_i and bridges β_j for $1 \leq i, j \leq k$.

Definition 8.20. Let $\mathcal{A}$ be a set of building blocks, and let

$$\mathcal{B} \subseteq \{\varphi + \varphi' \,:\, \varphi, \varphi' \in \mathcal{A}\}.$$

We consider the following two properties:

- (**B**) Whenever there is a permissible function $\varphi + \varphi' \in \mathcal{B}$ on γ_k such that $2D + \mathrm{div}(\varphi + \varphi')$ contains w_k, and either γ_k switches slope $\tau_k(\varphi)$ or $s_{k+1}(\varphi) < s'_k[\tau'_k(\varphi)]$, then there is some permissible function $\psi \in \mathcal{B}$ that agrees with $\varphi+\varphi'$ on $\Gamma_{\leq k}$ such that $s_{k+1}(\psi) > s_{k+1}(\varphi+\varphi')$.
- (**B′**) Whenever there are permissible functions in $\mathcal{B}$ that agree on γ_k with different slopes on β_{k+1}, and either γ_k is a switching loop or β_{k+1} is a switching bridge, then no permissible function in $\mathcal{B}$ is shiny on γ_k.

Theorem 8.21 says that we can construct a master template starting from any set of building blocks $\mathcal{A}$ that satisfies (**A**), and any collection $\mathcal{B}$ of pairwise sums of functions in $\mathcal{A}$ that satisfies (**B**) and (**B′**). In §9, we will choose such $\mathcal{A}$ and $\mathcal{B}$ on a case-by-case basis, according to the properties of Σ.

8.7. **The master template.** The rest of this section is dedicated to proving the following theorem.

Theorem 8.21. *Let $\mathcal{A}$ be a subset of the building blocks satisfying property* (**A**) *and let*

$$\mathcal{B} \subseteq \{\varphi + \varphi' \,:\, \varphi, \varphi' \in \mathcal{A}\}$$

be a subset satisfying properties (**B**) *and* (**B′**). *Then there is a tropical linear combination θ of the functions in $\mathcal{B}$ with $s_k(\theta)$ as specified above, such that*

- (i) *each function $\psi \in \mathcal{B}$ is assigned to some loop γ_k or bridge β_{k+1} and achieves the minimum on an open subset containing a point v of the loop or bridge to which it is assigned,*
- (ii) *any other function ψ' that achieves the minimum at v agrees with ψ on the loop γ_k.*

Note that, in the case where ψ and ψ' are assigned to the bridge β_{k+1}, the second condition says that they agree on the preceding loop γ_k.

8.8. **Algorithm for constructing the master template.** Throughout this section, and for the remainder of the paper, we assume that the hypotheses of Theorem 8.21 are satisfied. In particular, we let $\mathcal{A}$ be a subset of the building blocks satisfying property (**A**), and $\mathcal{B}$ a subset of pairwise sums of functions in $\mathcal{A}$ satisfying properties (**B**) and (**B′**).

We now sketch the overall procedure that we will use to build the master template

$$\theta = \min\{\psi + c_\psi \,:\, \psi \in \mathcal{B}\}$$

with the slopes $s_k(\theta)$ specified above. The algorithm is in many ways similar to that presented in §7, and we will highlight the differences when they appear. As in §7, we move from left to right across each of the three blocks where $s_k(\theta)$ is constant, adjusting the coefficients of unassigned permissible functions upward by smaller and smaller perturbations, and assigning functions to bridges and loops in such a way so that each function achieves the minimum on the bridge or loop to which it is assigned. At the end of each block, we start the next block by choosing coefficients such that θ bends at the midpoint of the bridge between blocks.

In the special case where the divisor class is vertex avoiding, the building blocks are the functions φ_i, the set $\mathcal{B}$ of all pairwise sums $\varphi_i + \varphi_j$ satisfies properties (**B**) and (**B′**) vacuously, and the template we construct is precisely the independence constructed in §7.

In the general case, our algorithm for constructing the master template is as follows.

Start at the First Bridge. If the multiplicity of β_1 is 2, then by Lemma 8.1, the multiplicity of all other loops and bridges is zero. Let $x \in \beta_1$ be a point with the property that the slopes $s_\zeta[i]$ are constant for all tangent vectors ζ in β_1 to the right of x. By restricting Σ to the subgraph to the right of x, we reduce to the vertex avoiding case, and we may employ the algorithm from §7. Otherwise, initialize the coefficient of every $\psi \in \mathcal{B}$ with $s_1(\psi) > 4$ so that it obtains the minimum somewhere on the first half of β_1, and assign all of them to β_1.

There are then several new permissible functions on the first loop. Initialize the coefficient of each new permissible function so that it equals θ halfway between the midpoint and the rightmost endpoint of β_1, and then apply the loop subroutine. Initialize all other coefficients to ∞. Proceed to the first loop.

Loop Subroutine. Each time we arrive at a loop γ_k, apply the following steps.

Loop Subroutine, Step 1: No unassigned permissible functions. If there are no unassigned permissible functions on γ_k, skip this loop and proceed to the next loop.

Loop Subroutine, Step 2: All unassigned permissible functions are new and agree on γ_k. If every unassigned permissible function on γ_k is new and in the same equivalence class, assign them all. Note that, by Lemma 8.11, if γ_k is not the first loop in a block, then all new functions are automatically in the same equivalence class. Set their coefficients so that they are equal to θ a short distance to the left of v_k. The slope of any new function along the bridge adjacent to v_k is smaller than the corresponding slope for a non-new function, and it follows that the new functions are the only functions to achieve the minimum in a neighborhood of v_k. Proceed to the next loop.

Loop Subroutine, Step 3: Re-initialize unassigned coefficients. Otherwise, there is at least one unassigned permissible function ψ on γ_k such that c_ψ is finite. Find the unassigned, permissible function $\psi \in \mathcal{B}$ that maximizes $\psi(w_k) + c_\psi$, among finite values of c_ψ. Initialize the coefficients of the new permissible functions (if any) and adjust the coefficients of the other unassigned permissible functions upward so that they are all equal to $\psi + c_\psi$ at w_k. The unassigned permissible functions are strictly less than all other functions, at every point in γ_k, even after this upward adjustment.

Loop Subroutine, Step 4: Assign departing functions. If there is a departing function, assign it to the loop. Note that any two departing functions agree on this loop, by Lemma 8.11. Adjust the coefficients of the non-departing unassigned permissible functions upward so that they are all equal to the departing function of smallest slope at a point on the following bridge a short distance to the right of w_k, but far enough so that the departing functions are the only functions to achieve the minimum at any point of the loop. This is possible because the building blocks have constant slopes along the bridges, and the bridges are much longer than the loops. Note that the departing functions achieve the minimum on the whole loop, no other functions achieve the minimum on this loop, and any two departing functions agree on the loop. Proceed to the next loop.

Loop Subroutine, Step 5: Skip skippable loops. In the vertex avoiding case, the loops with positive multiplicity are precisely the lingering loops, and they have the property that $s_k[i] = s_{k+1}[i]$ for all i. We skipped over these loops in the algorithm, without assigning a function. In the general case, the loops with unassigned permissible functions that we skip are characterized as follows.

Definition 8.22. We say that the loop γ_k is *skippable* if not all unassigned permissible functions are new and agree on γ_k, there are no unassigned departing permissible functions on γ_k, and there is an unassigned permissible function $\psi = \varphi + \varphi' \in \mathcal{B}$ satisfying one of the following:

(i) $2D + \mathrm{div}(\psi)$ contains a point whose distance from w_k is a non-integer multiple of m_k, or
(ii) $2D + \mathrm{div}(\psi)$ contains w_k, or
(iii) $D + \mathrm{div}(\varphi')$ contains two points of $\gamma_k \smallsetminus \{v_k\}$.

In the vertex avoiding case, conditions (ii) and (iii) of Definition 8.22 are never satisfied, and condition (i) is satisfied only on the lingering loops. Note that, if γ_k is skippable, then γ_k or β_{k+1}

has positive multiplicity. Also, whether a loop is skippable depends on which functions have been assigned. In particular, if there is an unassigned departing function, then the loop is not skippable.

If γ_k is skippable, then do not assign any functions. Proceed to the next loop.

Loop Subroutine, Step 6: Otherwise, use Proposition 8.24. In the remaining cases, when there are unassigned permissible functions, but not all are new, none are departing, and the loop is not skippable, we assign an equivalence class of permissible functions that agree on γ_k, chosen using the following lemma and proposition, which are close analogues of Lemma 7.18 and Proposition 7.20.

Lemma 8.23. *If there is no unassigned departing permissible function on γ_k, then the number of equivalence classes of permissible functions on γ_k is at most 3.*

Proof. Consider the set of building blocks $\varphi \in \mathcal{A}$ such that there exists $\varphi' \in \mathcal{A}$ with $\varphi + \varphi' \in \mathcal{B}$ an unassigned permissible function on γ_k. If every such building block satisfies $s_{k+1}(\varphi) = s'_k[\tau'_k(\varphi)]$, then the proof of Lemma 7.18 goes through essentially unchanged. Indeed, any two functions in $\mathcal{A}$ with the same s_{k+1} agree on γ_k by property (**A**). Thus, the number of equivalence classes of permissible functions on γ_k is bounded above by the number of pairs (i, j) such that $s'_k[i] + s'_k[j] = s_k(\theta)$. This number of pairs is at most 3, exactly as in the proof of Lemma 7.18.

On the other hand, suppose that there is an unassigned permissible function $\varphi + \varphi' \in \mathcal{B}$ such that $s_{k+1}(\varphi) < s'_k[\tau'_k(\varphi)]$. By property (**B**), there is a function $\psi \in \mathcal{B}$ that agrees with $\varphi + \varphi'$ on $\Gamma_{\leq k}$, with the property that

$$s_{k+1}(\psi) > s_{k+1}(\varphi + \varphi') \geq s_k(\theta).$$

Since ψ is a departing function, it must have been assigned to a previous loop. But this is impossible, because ψ agrees with $\varphi + \varphi'$ on all previous loops and bridges and $\varphi + \varphi'$ is unassigned. □

Proposition 8.24. *Consider a set of at most three equivalence classes of functions in $\mathcal{B}$ on a non-skippable loop γ_k. If all of the functions take the same value at w_k, then there is a point of γ_k at which one of these equivalence classes is strictly less than the others.*

Proof. The proof of Proposition 7.20 depends only on the restrictions of the functions to the loop γ_k, and the fact that, if ψ is one of these functions, then $2D + \mathrm{div}(\psi)$ does not contain w_k. This latter fact is guaranteed by our assumption that γ_k is not skippable. The conclusion therefore continues to hold if we replace the functions with equivalence classes of functions. □

Combining Lemma 8.23 and Proposition 8.24, we see that there is an equivalence class of unassigned permissible functions that achieves the minimum uniquely at some point of γ_k. We adjust the coefficients of these functions upward by $\frac{1}{3}m_k$. This ensures that they will never achieve the minimum on any loops to the right, yet still achieve the minimum uniquely on this loop. Assign all of the permissible functions in this equivalence class to the loop, and proceed to the next loop.

Proceeding to the Next Loop. If γ_k is not the last loop in its block, then apply the loop subroutine on γ_{k+1}. Otherwise, apply the following subroutine for proceeding to the next block.

Proceeding to the Next Block. After applying the loop subroutine to the last loop in a block, we will see that there is at most one equivalence class of unassigned permissible function, and these functions already achieve the minimum on the outgoing bridge, without any further adjustments of the coefficients. Assign them to the bridge.

If the current block is not the last one, then proceed to the first loop of the next block. There are several new permissible functions. Initialize the coefficient of each new permissible function so that it is equal to θ at the midpoint of the bridge between the blocks, and then apply the loop subroutine. Otherwise, we are at the last loop $\gamma_{g'}$, and proceed to the last bridge.

The Last Bridge. By Lemma 8.1, the ramification weight at $v_{g'+1}$ is at most 2, so $s_{g'+1}[1] = 1$ and $s_{g'+1}[0] = 0$. For each $\psi \in \mathcal{B}$ with $s_{g'+1}(\psi) = s_{g'+1}[0] + s_{g'+1}[1] = 1$, initialize the coefficient of ψ so that it equals θ at the midpoint of the last bridge, and assign ψ to $\beta_{g'+1}$. Similarly, for each $\psi' \in \mathcal{B}$ satisfying $s_{g'+1}(\psi') = 2s_{g'+1}[0] = 0$, initialize the coefficient of ψ' so that it it is equal to θ

halfway between the midpoint and the rightmost endpoint of the last bridge, and assign ψ' to $\beta_{g'+1}$. Output $\theta = \min_\psi\{\psi + c_\psi\}$.

Note that all functions assigned to a given loop agree on that loop. It is possible, however, for a function $\psi \in \mathcal{B}$ to be assigned to γ_k while another function ψ' that agrees with it is not, e.g., if ψ is departing and ψ' is not, or if ψ is permissible on γ_k and ψ' is not.

8.9. **Verifying the master template.** In this section, we prove Theorem 8.21, verifying that the master template constructed in the previous section has the claimed properties. We assume the hypotheses of the theorem; in particular, $\mathcal{A}$ is a set of building blocks that satisfies property (**A**) and $\mathcal{B}$ is a collection of pairwise sums of functions in $\mathcal{A}$ that satisfies (**B**) and (**B′**). We begin by checking that, if two functions in $\mathcal{B}$ are assigned to the first bridge β_1 and have the same slope on β_1, then the slope indices of their summands in $\mathcal{A}$ are the same.

Lemma 8.25. *Suppose $\varphi+\varphi'$, and $\phi+\phi'$ are elements of $\mathcal{B}$ such that $s_1(\varphi+\varphi') = s_1(\phi+\phi') > 4$. Then, after possibly reordering, we have $\tau'_0(\varphi) = \tau'_0(\phi)$ and $\tau'_0(\varphi') = \tau'_0(\phi')$.*

Proof. If $\mu(\beta_1) = 2$, then the verification is exactly as in the vertex avoiding case. Therefore, we may assume $\mu(\beta_1) \leq 1$. By Lemma 8.18, it suffices to show that, if

$$s'_0[i] + s'_0[i'] = s'_0[j] + s'_0[j'] > 4,$$

then $(i, i') = (j, j')$. By (29), our hypotheses on the ramification sequence of V at p imply that either $s'_0[5] \leq 2$ or $s'_0[4] + s'_0[6] \leq 5$. If $s'_0[5] \leq 2$, then $s'_0[i] + s'_0[j] \leq 4$ for all pairs $i \leq i' \leq 5$. Note that $s'_0[6] + s'_0[i] = s'_0[6] + s'_0[j]$ if and only if $i = j$, so the conclusion follows in this case.

On the other hand, if $s'_0[5] \geq 3$ and $s'_0[4] + s'_0[6] \leq 5$, then $s'_0[i] + s'_0[i'] \leq 4$ for all pairs $i < i' \leq 5$. It therefore suffices to show that there is no i such that $s'_0[i] + s'_0[6] = 2s'_0[5]$. By assumption, however, we have

$$s'_0[5] + s'_0[6] > 2s'_0[5] \geq 6,$$

which is greater than $s'_0[4] + s'_0[6] \geq s'_0[i] + s'_0[6]$ for all $i < 4$. □

We have the following straightforward generalization of Lemma 7.21.

Lemma 8.26. *Suppose that ψ is assigned to the loop γ_k or the bridge β_k, and let $k' > k$ be the smallest value such that there is a function assigned to $\gamma_{k'}$. Then ψ does not achieve the minimum at any point to the right of $v_{k'}$.*

Proof. This follows by the same argument as Lemma 7.21, using the definition of permissibility and the fact that the building blocks have constant slopes along bridges. □

Next, we prove a generalization of Lemma 7.22: on each non-skippable loop, there is a point where the function assigned to that loop achieves the minimum, and the other functions that achieve the minimum agree on the loop.

Lemma 8.27. *Suppose that ψ is assigned to the loop γ_k. Then there is a point $v \in \gamma_k$ where ψ achieves the minimum. Moreover, any other function ψ' that achieves the minimum at v agrees with ψ on γ_k.*

Proof. This follows by the same argument as Lemma 7.22. □

The analogous statement about functions assigned to bridges will be proved by a counting argument similar to that in the proof of Theorem 7.1; see the proof of Theorem 8.21.

The remainder of this section is devoted to showing that every function in $\mathcal{B}$ is assigned to some loop or bridge. Ultimately, this is a counting argument similar to that of §7.4, but the details are more subtle. We proceed via a sequence of lemmas and propositions. The following two propositions are analogues of Lemmas 7.11 and 7.15, respectively, with shiny functions in place of new permissible functions.

Proposition 8.28. *If γ_k is skippable and not the first loop in a block, then no permissible function is shiny on γ_k.*

Proposition 8.29. *The loops $\gamma_z, \gamma_b, \gamma_{b'}$, and $\gamma_{z'+2}$ are all non-skippable, and no permissible function is shiny on any of them.*

The proofs of these propositions rely heavily on property (**B**), and use the following two technical lemmas about permissible functions on skippable loops.

Lemma 8.30. *Let γ_k be a skippable loop and let $\psi = \varphi + \varphi' \in \mathcal{B}$ be an unassigned permissible function on γ_k. Then*

(i) $s_{k+1}(\psi) = s_k(\theta)$,
(ii) $s_{k+1}(\varphi) = s'_k[\tau'_k(\varphi)]$, *and*
(iii) $s_{k+1}(\varphi') = s'_k[\tau'_k(\varphi')]$.

Proof. By definition, no unassigned permissible function is departing on a skippable loop. Therefore,

$$s_{k+1}(\psi) = s_k(\theta).$$

It remains to show that $s_{k+1}(\varphi) = s'_k[\tau'_k(\varphi)]$. Suppose not. Then

$$s_{k+1}(\varphi) < s'_k[\tau'_k(\varphi)],$$

and, by property (**B**), there is a function $\psi' \in \mathcal{B}$ that agrees with ψ on $\Gamma_{\leq k}$, with the property that

$$s_{k+1}(\psi') > s_{k+1}(\psi) = s'_k(\theta).$$

We claim that this is impossible. Indeed, if ψ' is unassigned on γ_k, then it would be a departing function, contradicting the hypothesis that γ_k is skippable. On the other hand, if ψ' is assigned to a previous loop then ψ would have been assigned to that loop as well.

We conclude that $s_{k+1}(\varphi) = s'_k[\tau'_k(\varphi)]$, and, similarly, $s_{k+1}(\varphi') = s'_k[\tau'_k(\varphi')]$, as required. □

Lemma 8.31. *Suppose γ_k is a skippable loop, and there is a building block ϕ such that $s_{k+1}(\phi) > s_k[\tau_k(\phi)]$. Then there is a permissible function $\psi = \varphi + \varphi' \in \mathcal{B}$ such that*

(i) $s_{k+1}(\varphi) = s_k(\varphi) + 1$,
(ii) $s_{k+1}(\varphi') = s_k(\varphi') - 1$, *and*
(iii) $D + \mathrm{div}(\varphi')$ *contains either w_k, a point of γ_k whose distance from w_k is not an integer multiple of m_k, or two points of $\gamma_k \smallsetminus \{v_k\}$.*

Proof. Since γ_k is skippable, there is an unassigned permissible function $\psi = \varphi + \varphi' \in \mathcal{B}$ such that $2D + \mathrm{div}(\psi)$ contains either w_k, a point whose distance from w_k is a non-integer multiple of m_k, or two points of $\gamma_k \smallsetminus \{v_k\}$. We first consider the case where one of the two functions φ, φ' has smaller slope on β_{k+1} than on β_k. Suppose without loss of generality that

$$s_{k+1}(\varphi') < s_k(\varphi').$$

Since ψ is permissible, we must have $s_{k+1}(\psi) \geq s_k(\psi)$. It follows that $s_{k+1}(\varphi) > s_k(\varphi)$. Since the slope of a building block can increase by at most 1 from one bridge to the next, we see that

$$s_{k+1}(\varphi) = s_k(\varphi) + 1 \text{ and } s_{k+1}(\varphi') = s_k(\varphi') - 1.$$

It follows that the restriction of $D + \mathrm{div}(\varphi)$ to γ_k is zero, and $D + \mathrm{div}(\varphi')$ contains either w_k, a point of γ_k whose distance from w_k is not an integer multiple of m_k, or two points of $\gamma_k \smallsetminus \{v_k\}$.

To complete the proof, we will rule out the possibility that neither function φ, φ' has smaller slope on β_{k+1} than on β_k. Assume that

$$s_{k+1}(\varphi) \geq s_k(\varphi) \text{ and } s_{k+1}(\varphi') \geq s_k(\varphi').$$

Note that this immediately rules out the possibility that $D + \mathrm{div}(\varphi')$ contains more than one point of γ_k. We will reach a contradiction by showing that γ_k is a switching loop and then applying property (**B**). As a first step in this direction, we claim that $2D + \mathrm{div}\,\psi$ contains w_k. Since

$s_{k+1}(\phi) > s_k[\tau_k(\phi)]$, the restriction of $D+\mathrm{div}(\phi)$ to $\gamma_k \smallsetminus \{v_k\}$ has degree 0 (see, e.g., Remark 8.15). It follows that the shortest distance from the point of D on γ_k to w_k is an integer multiple of m_k. Combined with our assumption that the slopes of φ and φ' do not decrease from β_k to β_{k+1}, we see that the shortest distances from w_k to the point of $D+\mathrm{div}(\varphi)$ and the point of $D+\mathrm{div}(\varphi')$ on γ_k are integer multiples of m_k as well. Therefore, $2D+\mathrm{div}(\psi)$ cannot contain a point whose shortest distance from w_k is a non-integer multiple of m_k, and must therefore contain w_k, as claimed.

Without loss of generality, we assume that $D+\mathrm{div}(\varphi)$ contains w_k. Since the restriction of $D+\mathrm{div}(\varphi)$ to γ_k has degree at most 1, we see that this restriction is equal to w_k. It follows that

$$s_{k+1}(\varphi) = s_k(\varphi),$$

and φ agrees with ϕ on γ_k, but $s_{k+1}(\varphi) = s_{k+1}(\phi) - 1$.

We now show that γ_k switches slope $\tau_k(\phi)$. By assumption, $s_{k+1}(\phi) > s_k[\tau_k(\phi)]$. Combining this with the two equations above, we see that

$$s_k(\varphi) \geq s_k[\tau_k(\phi)].$$

This implies $\tau_k(\varphi) \geq \tau_k(\phi)$. By Lemma 8.30, however, we have

$$s_{k+1}(\varphi) = s'_k[\tau'_k(\varphi)],$$

so $\tau'_k(\varphi) < \tau'_k(\phi)$. Combining these inequalities with the fact that slope index sequences are nondecreasing, we see that

$$\tau_k(\phi) \leq \tau_k(\varphi) \leq \tau'_k(\varphi) < \tau'_k(\phi).$$

Since ϕ is a building block, by Definition 8.3(i), it follows that γ_k switches slope $\tau_k(\phi)$.

We now apply property (**B**) again, in a similar way to the proof of Lemma 8.30. Specifically, there is a function $\psi' \in \mathcal{B}$ that agrees with ψ on $\Gamma_{\leq k}$, with the property that

$$s_{k+1}(\psi') > s_{k+1}(\psi) = s'_k(\theta).$$

If ψ' has not been assigned to a previous loop, then it is an unassigned departing function on γ_k, and for this reason γ_k is not skippable. If ψ' has been assigned to a previous loop, then since ψ agrees with ψ' on this previous loop, we see that ψ must have been assigned to this previous loop as well. We therefore arrive at a contradiction, which rules out the possibility that neither φ nor φ' has smaller slope on β_{k+1} than on β_k and completes the proof of the lemma. □

Proof of Proposition 8.28. By Proposition 8.14, any shiny permissible function on γ_k has a summand $\phi \in \mathcal{A}$ satsifying $s_{k+1}(\phi) > s_k[\tau_k(\phi)]$. We will assume that such a function ϕ exists, and consider unassigned permissible functions of the form $\phi + \phi' \in \mathcal{B}$. Since ϕ exists, by Lemma 8.31, there is an unassigned permissible function $\varphi + \varphi' \in \mathcal{B}$ such that

$$s_{k+1}(\varphi) = s_k(\varphi) + 1, \quad s_{k+1}(\varphi') = s_k(\varphi') - 1,$$

and $D+\mathrm{div}(\varphi')$ contains either w_k, or a point of γ_k whose distance from w_k is not an integer multiple of m_k, or two points of $\gamma_k \smallsetminus \{v_k\}$.

Both $D+\mathrm{div}(\phi)$ and $D+\mathrm{div}(\varphi)$ contain no points of $\gamma_k \smallsetminus \{v_k\}$. Any two building blocks with this property agree, and have the same slope along the bridge β_{k+1}. It follows that $s_{k+1}(\varphi) = s_{k+1}(\phi)$. By Lemma 8.30, we also have

$$s_{k+1}(\varphi) + s_{k+1}(\varphi') = s_{k+1}(\phi) + s_{k+1}(\phi') = s_k(\theta).$$

Subtracting, we see that $s_{k+1}(\varphi') = s_{k+1}(\phi')$. Moreover, by Lemma 8.30, we have

$$s_{k+1}(\varphi') = s'_k[\tau'_k(\varphi')] \text{ and } s_{k+1}(\phi') = s'_k[\tau'_k(\phi')],$$

so $\tau'_k(\varphi') = \tau'_k(\phi')$, which, by property (**A**), implies that φ' and ϕ' agree on γ_k.

Since φ' and ϕ' agree on γ_k and their slopes on β_{k+1} are equal, the difference between $\mathrm{div}\,\varphi'$ and $\mathrm{div}\,\phi'$ on γ_k must be supported at v_k. Now the restriction of $D+\mathrm{div}(\varphi')$ to γ_k has degree 2 and, since $\phi+\phi'$ is shiny, the restriction of $2D+\mathrm{div}(\phi+\phi')$ to $\gamma_k \smallsetminus \{v_k\}$ has degree at most 1. It follows that $D+\mathrm{div}(\varphi')$ contains v_k. This forces $D+\mathrm{div}(\varphi')$ to be supported at points whose shortest distance to w_k is an integer multiple of m_k. Recall, however, that φ' was chosen so that $D+\mathrm{div}(\varphi')$ contains either w_k or a point of γ_k whose distance from w_k is not an integer multiple of m_k. We therefore see that $D+\mathrm{div}(\varphi')$ contains w_k, and hence

$$[D+\mathrm{div}(\varphi')]_{|\gamma_k} = v_k + w_k.$$

Thus, φ' agrees with φ and ϕ on γ_k, but since $D+\mathrm{div}(\varphi')$ contains v_k and $D+\mathrm{div}(\varphi)$ does not, we have $s_k(\varphi) < s_k(\varphi')$. It follows that $\tau_k(\varphi) \leq \tau_k(\varphi')$. Similarly, since $D+\mathrm{div}(\varphi')$ contains w_k and $D+\mathrm{div}(\varphi)$ does not, we have

$$s_{k+1}(\varphi) > s_{k+1}(\varphi').$$

By Lemma 8.30, however, we have

$$s_{k+1}(\varphi) = s'_k[\tau'_k(\varphi)] \text{ and } s_{k+1}(\varphi') = s'_k[\tau'_k(\varphi')].$$

Thus, $\tau'_k(\varphi) > \tau'_k(\varphi')$. Combining these inequalities with the fact that slope index sequences are nondecreasing, we see that

$$\tau_k(\varphi) \leq \tau_k(\varphi') \leq \tau'_k(\varphi') < \tau'_k(\varphi).$$

Since φ is a building block, by Definition 8.3(i), it follows that γ_k switches slope $\tau_k(\varphi)$.

By property (**B**), there is a function $\psi' \in \mathcal{B}$ that agrees with $\phi+\phi'$ on $\Gamma_{\leq k}$, with the property that

$$s_{k+1}(\psi') > s_{k+1}(\phi+\phi') = s'_k(\theta).$$

If ψ' has not been assigned to a previous loop, then it is an unassigned departing function on γ_k, and for this reason γ_k is not skippable. If ψ' has been assigned to a previous loop, then since $\phi+\phi'$ agrees with ψ' on this previous loop, we see that $\phi+\phi'$ must have been assigned to this previous loop as well. This contradicts our assumption that $\phi+\phi'$ was unassigned. We conclude that there are no shiny functions on γ_k, as required. □

Proof of Proposition 8.29. Let $k \in \{z, b, b', z'+2\}$, and note that the choice of these four loops γ_k guarantees that there is an index i such that $s_k[i] < s'_k[i]$. We must show that γ_k is not skippable, and that no permissible function is shiny on γ_k. We begin by showing that γ_k is not skippable.

Suppose γ_k is skippable. Then there is an unassigned permissible function $\psi = \varphi+\varphi'$ on γ_k such that $2D+\mathrm{div}(\varphi+\varphi')$ contains either w_k or a point whose distance from w_k is not an integer multiple of m_k, or $D+\mathrm{div}(\varphi')$ contains two points of $\gamma_k \smallsetminus \{v_k\}$. By Lemma 8.30, we have

$$s_{k+1}(\varphi) + s_{k+1}(\varphi') = s'_k[\tau'_k(\varphi)] + s'_k[\tau'_k(\varphi')] = s_k(\theta).$$

Moreover, by Lemma 8.31, we have

$$s_{k+1}(\varphi) = s_k(\varphi) + 1,$$

hence the restriction of $D+\mathrm{div}(\varphi)$ to γ_k must have degree 0, which forces $\tau'_k(\varphi) = i$. As in Lemma 7.15, the choice of z, b, b', and z' ensures that there does not exist a value of j such that $s'_k[i] + s'_k[j] = s'_k(\theta)$. Thus, γ_k cannot be a skippable loop.

It remains to show that there are no shiny permissible functions on γ_k. Let φ be a function satisfying $s_k(\varphi) = s_k[i]$ and $s_{k+1}(\varphi) = s'_k[i]$. Any function that is shiny on γ_k must agree with a function of the form $\psi = \varphi + \varphi'$ with $s_{k+1}(\varphi+\varphi') = s_k(\theta)$. Again, because there is no j such that $s'_k[i] + s'_k[j] = s'_k(\theta)$, we see that $s_{k+1}(\varphi')$ cannot equal $s'_k[j]$ for any j. This means that

$$s_{k+1}(\varphi') < s'_k[\tau'_k(\varphi')].$$

Hence, $D+\mathrm{div}(\varphi')$ contains w_k. Since ψ is shiny, the restriction of $2D+\mathrm{div}(\psi)$ to $\gamma_k \smallsetminus \{v_k\}$ has degree at most 1, so this restriction is exactly w_k. It follows that ψ agrees with 2φ on γ_k, and

$$s_{k+1}(\psi) = s_k(\theta) = s_{k+1}(2\varphi) - 1.$$

This implies that $s_k(\theta)$ is odd, so γ_k is contained in the middle block, and $s_k(\theta) = 3$. However, b and b' were chosen so that $s_{k+1}(\varphi)$ is at most 1 if the box contained in λ'_k but not λ_k is in the first row, and $s_{k+1}(\varphi)$ is at least 3 if this box is contained in the second or third row. □

We now analyze the output of the algorithm. We first note the following.

Lemma 8.32. *Suppose that $\varphi+\varphi'$ and $\phi+\phi'$ are assigned to the same loop γ_k. Then, after possibly reordering the summands, φ agrees with ϕ and φ' agrees with ϕ' on γ_k. Moreover,*

$$\tau_k(\varphi) = \tau_k(\phi), \quad \tau_k(\varphi') = \tau_k(\phi'), \quad \tau'_k(\varphi) = \tau'_k(\phi) \quad \textit{and} \quad \tau'_k(\varphi') = \tau'_k(\phi').$$

Proof. Since both functions are assigned to the loop γ_k, we see that $\varphi + \varphi'$ agrees with $\phi + \phi'$ on γ_k. It suffices to show that φ agrees with ϕ on γ_k. Indeed, if φ agrees with ϕ on γ_k, then the fact that $\varphi + \varphi'$ agrees with $\phi + \phi'$ will then imply that φ' agrees with ϕ', and Remark 8.2 shows the equality of slope indices.

The fact that $\varphi + \varphi'$ agrees with $\phi + \phi'$ is equivalent to the statement that the restrictions of $2D + \mathrm{div}(\varphi + \varphi')$ and $2D + \mathrm{div}(\phi + \phi')$ to $\gamma_k \smallsetminus \{v_k, w_k\}$ are the same. Thus, if $D + \mathrm{div}(\varphi)$ contains a point $v \in \gamma_k \smallsetminus \{v_k, w_k\}$, then one of $D + \mathrm{div}(\phi)$ or $D + \mathrm{div}(\phi')$ must contain v as well. If v is the only point of $\gamma_k \smallsetminus \{v_k, w_k\}$ contained in both $D + \mathrm{div}(\varphi)$ and $D + \mathrm{div}(\phi)$, then φ and ϕ agree on γ_k. It follows that, if the restrictions of $D + \mathrm{div}(\varphi), D + \mathrm{div}(\varphi'), D + \mathrm{div}(\phi)$, and $D + \mathrm{div}(\phi')$ to $\gamma_k \smallsetminus \{v_k, w_k\}$ all have degree at most 1, then the conclusion holds.

The other possibility is that the restriction of one of these 4 divisors to $\gamma_k \smallsetminus \{v_k, w_k\}$ has degree 2. However, this implies that either γ_k is skippable, or the assigned functions are departing. If both $\varphi+\varphi'$ and $\phi+\phi'$ are departing, then the restrictions to γ_k of $2D+\mathrm{div}(\varphi+\varphi')$ and $2D+\mathrm{div}(\phi+\phi')$ have degree at most 1, contradicting our assumption that one of them has degree 2. Since $\varphi + \varphi'$ and $\phi + \phi'$ are assigned to γ_k, the loop cannot be skippable. □

Recall that, in the vertex avoiding case, Corollary 7.16 says that the number of permissible functions on a block is exactly 1 more than the number of non-lingering loops in that block. This was shown by counting the number of permissible functions on the first loop of the block, and then observing that there is at most one new permissible function on every non-lingering loop. Since there are no new permissible functions on lingering loops, the same observation shows that the number of unassigned permissible functions never increases, when proceeding from one loop γ_k to the next in a block, and that it decreases by one when $k \in \{z, b, b', z' + 2\}$.

In the general case, we may assign several functions to the same loop, so instead of counting individual unassigned permissible functions, we count collections of such functions, which we call *cohorts* and define as follows.

Definition 8.33. We say that a function $\psi \in \mathcal{B}$ *leaves its shine* on the last loop γ_k satisfying:

(i) ψ is shiny or new on γ_k, and
(ii) ψ is not assigned to a loop $\gamma_{k'}$ with $k' < k$.

Two unassigned permissible functions are in the same *cohort* on γ_ℓ if they both leave their shine and agree on some loop γ_k, with $k \leq \ell$.

Every function is new on the first loop where it is permissible. Then, eventually, there is a loop where it leaves its shine and joins a cohort. On any loop that is not the first loop in a block, all shiny or new functions agree, so at most one new cohort is created. This is one way in which new cohorts behave like the new permissible functions in §7.4; there may be several new cohorts on the first loop of a block, and then at most one new cohort on each subsequent loop. Furthermore, there are no shiny functions on γ_k, for $k \in \{z, b, b', z' + 2\}$, so no new cohorts are formed on these loops. The next proposition says that, on each loop where a new cohort is created, and also on the special loops γ_k, for $k \in \{z, b, b', z' + 2\}$, an entire cohort is assigned. This is essential for the proof of Theorem 8.21, where we bound the number of cohorts on each loop while moving from left to right across a block to show that every function in $\mathcal{B}$ is assigned to some loop or bridge.

Remark 8.34. On a non-skippable loop where no new cohort is created, the functions that are assigned typically form a proper subset of a cohort. In this way, cohorts may lose members as we move from left to right across a block, but the number of cohorts on each loop behaves just as predictably as the number of unassigned permissible functions in the vertex avoiding case.

In the vertex avoiding case, each cohort consists of a single unassigned permissible function, and the argument for counting cohorts in the proof of Theorem 8.21 specializes to the argument for counting permissible functions in §7.4.

Proposition 8.35. *Suppose that some function leaves its shine on γ_ℓ, or $\ell \in \{z, b, b', z'+2\}$. If $\psi \in \mathcal{B}$ is assigned to γ_ℓ, then any other function in the same cohort on γ_ℓ is also assigned to γ_ℓ.*

To prove Proposition 8.35, we will use property ($\mathbf{B'}$) along with a preliminary lemma.

Lemma 8.36. *Let $\psi = \varphi + \varphi'$ and $\psi' = \phi + \phi'$ be functions that leave their shine and agree on γ_{k_0}. Suppose k is the smallest integer such that $k \geq k_0$ and the sets of slope indices $\{\tau_{k+1}(\varphi), \tau_{k+1}(\varphi')\}$ and $\{\tau_{k+1}(\phi), \tau_{k+1}(\phi')\}$ are different. Suppose, furthermore, that ψ and ψ' are unassigned and permissible on γ_k. Then*

(i) *either γ_k is a switching loop or β_{k+1} is a switching bridge,*
(ii) *$k \notin \{z, b, b', z'+2\}$, and*
(iii) *either ψ or ψ' is assigned to γ_k.*

Furthermore, if one of ψ, ψ' is assigned to γ_k and the other is not, then no function is shiny on γ_k.

Proof. By assumption, ψ agrees with ψ' on γ_{k_0}. Also, by Lemma 8.16, we have that the sets of slope indices $\{\tau_{k_0}(\varphi), \tau_{k_0}(\varphi')\}$ and $\{\tau_{k_0}(\phi), \tau_{k_0}(\phi')\}$ are the same. Lemma 8.19 then says that ψ agrees with ψ' on γ_t for all t in the range $k_0 \leq t < k$.

After possibly relabeling the functions, we may assume that $\tau_k(\varphi) = \tau_k(\phi)$ and $\tau_k(\varphi') = \tau_k(\phi')$, and suppose $\tau_{k+1}(\varphi) > \tau_{k+1}(\phi)$. Since slope indices of building blocks only change due to switching (Definition 8.3), it follows that either γ_k switches slope $\tau_k(\phi)$ or β_{k+1} switches slope $\tau'_k(\phi)$. It follows that $k \notin \{z, b, b', z'+2\}$. Since a switching loop or bridge can switch at most one slope, we also have $\tau_{k+1}(\varphi') \geq \tau_{k+1}(\phi')$, with equality if $\tau_k(\varphi) \neq \tau_k(\varphi')$.

Claim 1: Either ψ or ψ' is assigned to γ_k. Suppose that neither ψ nor ψ' is assigned to γ_k. Then ψ and ψ' are both permissible and not shiny on γ_{k+1}. Therefore,

$$s_{k+1}(\varphi) = s_{k+1}[\tau_{k+1}(\varphi)], \quad s_{k+1}(\varphi') = s_{k+1}[\tau_{k+1}(\varphi')],$$
$$s_{k+1}(\phi) = s_{k+1}[\tau_{k+1}(\phi)], \quad s_{k+1}(\phi') = s_{k+1}[\tau_{k+1}(\phi')].$$

Summing these, we obtain

$$s_{k+1}(\psi) = s_{k+1}[\tau_{k+1}(\varphi)] + s_{k+1}[\tau_{k+1}(\varphi')] > s_{k+1}[\tau_{k+1}(\phi)] + s_{k+1}[\tau_{k+1}(\phi')] = s_{k+1}(\psi').$$

It follows that ψ is departing on γ_k. This contradicts the supposition that neither ψ nor ψ' is assigned to γ_k and proves the claim.

It remains to show that if one of ψ, ψ' is not assigned to γ_k, then no function is shiny on γ_k.

Claim 2: If ψ and ψ' agree on γ_k, then no function is shiny. This is straightforward. Indeed, if ψ and ψ' agree and one is assigned while the other is not, then the one that is assigned is departing and the other is not. In this case, no function is shiny on γ_k by property ($\mathbf{B'}$).

For the remainder of the proof of the lemma, we assume ψ and ψ' do not agree on γ_k, and show that no function is shiny.

Claim 3: The functions φ and ϕ do not agree on γ_k. Since ψ and ψ' do not agree on γ_k, either φ and ϕ do not agree on γ_k, or φ' and ϕ' do not agree on γ_k. By property ($\mathbf{A}$), if φ' and ϕ' do not agree, then $\tau'_k(\varphi') \neq \tau'_k(\phi')$. This implies that γ_k switches slope $\tau'_k(\varphi')$. Since a switching loop can switch at most one slope, it follows that φ' agrees with φ on γ_k, $\tau_k(\varphi') = \tau_k(\varphi)$, and $\tau_{k+1}(\varphi') = \tau_{k+1}(\varphi)$. In this case, we may relabel φ and φ' without loss of generality, and the result follows.

Claim 4: The loop γ_k is a decreasing loop and switches slope $\tau_k(\phi)$. To see this, note that neither ψ nor ψ' is shiny on γ_k, hence

$$s_k(\varphi) = s_k[\tau_k(\varphi)], \quad s_k(\varphi') = s_k[\tau_k(\varphi')], \quad s_k(\phi) = s_k[\tau_k(\phi)], \quad \text{and} \quad s_k(\phi') = s_k[\tau_k(\phi')].$$

It follows that $s_k(\varphi) = s_k(\phi)$ and $s_k(\varphi') = s_k(\phi')$.

Recall that, on a loop, every divisor of degree 1 is equivalent to a unique effective divisor. Thus, since φ and ϕ have the same incoming slope, if the restrictions of $D + \mathrm{div}(\varphi)$ and $D + \mathrm{div}(\phi)$ to $\gamma_k \smallsetminus \{w_k\}$ each have degree at most 1, then φ agrees with ϕ on γ_k. Because we showed, in the previous claim, that φ and ϕ do not agree on γ_k, the restriction of $D + \mathrm{div}(\phi)$ to $\gamma_k \smallsetminus \{w_k\}$ has degree 2. Equivalently,

$$s_{k+1}(\phi) = s_k(\phi) - 1,$$

so γ_k is a decreasing loop and hence has positive multiplicity. We already showed that either γ_k switches slope $\tau_k(\phi)$ or β_k is a switching bridge. Since switching bridges have multiplicity 2 and the sum of all multiplicities is at most 2, it follows that γ_k switches slope $\tau_k(\phi)$, as claimed.

We now complete the proof that no function is shiny on γ_k. Suppose $\eta + \eta'$ is shiny on γ_k. By Proposition 8.14, after possibly relabeling, the restriction of $D + \mathrm{div}(\eta)$ to $\gamma_k \smallsetminus \{v_k\}$ has degree 0. Because ψ' is permissible on γ_k, and $s_{k+1}(\phi) < s_k(\phi)$, as shown above, we must have

$$s_{k+1}(\phi') = s_k(\phi') + 1.$$

Thus the restriction of $D + \mathrm{div}(\phi')$ to $\gamma_k \smallsetminus \{v_k\}$ also has degree 0, η agrees with ϕ' on γ_k, and

$$s_{k+1}(\eta') = s_{k+1}(\phi').$$

Since $\eta + \eta'$ is permissible on γ_k, we have $s_{k+1}(\eta) \geq s_{k+1}(\phi)$. Also, since $\eta + \eta'$ is shiny,

$$s_k[\tau_k(\eta)] + s_k[\tau_k(\eta')] < s_k(\theta) = s_k[\tau_k(\phi)] + s_k[\tau_k(\phi')].$$

It follows that either $\tau_k(\eta) < \tau_k(\phi)$, and γ_k switches slope $\tau_k(\eta)$, or $\tau_k(\eta') < \tau_k(\phi')$, and γ_k switches slope $\tau_k(\eta')$. However, we will show that neither of these is possible. Indeed, the first is impossible because γ_k switches slope $\tau_k(\phi)$, and a loop can switch at most 1 slope. The second requires

$$\tau_k(\eta') = \tau_k(\phi') - 1 = \tau_k(\phi).$$

However, since

$$s_k[\tau_k(\eta')] < s_k[\tau_k(\phi')] < s'_k[\tau'_k(\phi')],$$

we see that $s_k[\tau_k(\eta')] \leq s'_k[\tau_k(\eta') + 1] + 2$. Since the slope of a function in $R(D)$ can increase by at most one from the left side of γ_k to the right side, we see that there is no function η'' with $s_k(\eta'') \leq s_k[\tau_k(\eta')]$ and $s'_k(\eta'') \geq s'_k[\tau_k(\eta') + 1]$. This shows that it is impossible for γ_k to switch slope $\tau_k(\phi') - 1$, and completes the proof of the lemma. ☐

Proof of Proposition 8.35. Suppose that ψ and ψ' are in the same cohort on γ_ℓ, and that the functions assigned to γ_ℓ include ψ but not ψ'. We must show that $\ell \notin \{z, b, b', z' + 2\}$ and no function leaves its shine on γ_ℓ.

Let γ_{k_0} be the loop where ψ and ψ' leave their shine. By Lemma 8.36, if the set of slope indices $\{\tau_k(\varphi), \tau_k(\varphi')\}$ and $\{\tau_k(\phi), \tau_k(\phi')\}$ are different for some $k_0 \leq k \leq \ell$, then one of ψ or ψ' is assigned to γ_k, contradicting our assumption that they are in the same cohort on γ_ℓ. Furthermore, if $\{\tau_{\ell+1}(\varphi), \tau_{\ell+1}(\varphi')\}$ and $\{\tau_{\ell+1}(\phi), \tau_{\ell+1}(\phi')\}$ are different, then by Lemma 8.36, $\ell \notin \{z, b, b', z' + 2\}$ and no function leaves its shine on γ_ℓ. We may therefore assume that the sets of slope indices $\{\tau_k(\varphi), \tau_k(\varphi')\}$ and $\{\tau_k(\phi), \tau_k(\phi')\}$ are the same for $k_0 \leq k \leq \ell + 1$. By Lemma 8.19, we then have

$$\tau'_\ell(\varphi) = \tau'_\ell(\phi) \text{ and } \tau'_\ell(\varphi') = \tau'_\ell(\phi')$$

and both pairs of functions agree on γ_ℓ by property (**A**).

Since ψ and ψ' agree on γ_ℓ, and the functions assigned to γ_ℓ include ψ but not ψ', we see that ψ is a departing function, but ψ' is not. In this case, since ψ' leaves its shine on γ_{k_0}, we see that ψ' is not shiny on $\gamma_{\ell+1}$, hence we must have

$$s_{\ell+1}(\phi) = s_{\ell+1}[\tau_{\ell+1}(\phi)] \text{ and } s_{\ell+1}(\phi') = s_{\ell+1}[\tau_{\ell+1}(\phi')].$$

We show that $\ell \notin \{z, b, b', z'+2\}$. If γ_ℓ is a switching loop or β_ℓ is a switching bridge, then $\ell \notin \{z, b, b', z'+2\}$, hence we may assume that $\tau_\ell = \tau'_\ell = \tau_{\ell+1}$ for each of the functions φ, φ', ϕ, and ϕ'. Because ψ is departing, we have either

$$s'_\ell[\tau'_\ell(\varphi)] > s_\ell[\tau_\ell(\varphi)] \text{ or } s'_\ell[\tau'_\ell(\varphi)] > s_\ell[\tau_\ell(\varphi)].$$

Assume without loss of generality that the first inequality holds. If $s_{\ell+1}[\tau_{\ell+1}(\varphi)] < s'_\ell[\tau'_\ell(\varphi)]$, then $\lambda_{\ell+1}$ is contained in λ_ℓ, hence $\ell \notin \{z, b, b', z'+2\}$. Suppose $s_{\ell+1}[\tau_{\ell+1}(\varphi)] = s'_\ell[\tau'_\ell(\varphi)]$. Since ψ' is not departing, we have

$$s_\ell(\theta) = s'_\ell(\psi') = s'_\ell[\tau'_{\ell+1}(\varphi)] + s'_\ell[\tau'_\ell(\varphi')].$$

But z, b, b', and z' are chosen so that there is no integer j such that $s_\ell(\theta) = s'_\ell[\tau'_\ell(\phi)] + s'_\ell[j]$, so again $\ell \notin \{z, b, b', z'+2\}$.

It remains to show that no function leaves its shine on γ_ℓ. Because ψ is departing, either φ or φ' must have higher slope on $\beta_{\ell+1}$ than on β_ℓ. Without loss of generality we may assume that $s_{\ell+1}(\varphi) > s_\ell(\varphi)$. Our assumption that the slope indices of φ and φ' agree with those of ϕ and ϕ' implies that either

$$s_{\ell+1}(\varphi) > s_{\ell+1}[\tau_{\ell+1}(\varphi)] \text{ or } s_{\ell+1}(\varphi') > s_{\ell+1}[\tau_{\ell+1}(\varphi')].$$

In other words, either

$$s'_\ell[\tau'_\ell(\varphi)] > s_{\ell+1}[\tau'_\ell(\varphi)] \text{ or } s'_\ell[\tau'_\ell(\varphi')] > s_{\ell+1}[\tau'_\ell(\varphi')].$$

Now, assume that $\eta+\eta' \in \mathcal{B}$ is shiny on γ_ℓ. In order to show that $\eta+\eta'$ does not leave its shine on γ_ℓ, we first show that $\eta+\eta'$ cannot agree with a departing function on γ_ℓ. Any function that is both departing and shiny on γ_ℓ agrees with 2φ, and such a function exists only if $s_{\ell+1}(2\varphi) = s_\ell(\theta)+1$. From this we see that both

$$s_{\ell+1}(\varphi) > s_{\ell+1}[\tau_{\ell+1}(\varphi)] \text{ and } s_{\ell+1}(\varphi') > s_{\ell+1}[\tau_{\ell+1}(\varphi')].$$

But, because

$$s_{\ell+1}(\phi) = s_{\ell+1}[\tau_{\ell+1}(\phi)] \text{ and } s_{\ell+1}(\phi') = s_{\ell+1}[\tau_{\ell+1}(\phi')],$$

we see that $s_{\ell+1}(\phi+\phi') = s_\ell(\theta)-1$. This implies that $\phi+\phi'$ is not permissible on γ_ℓ, a contradiction.

We now show that $\eta+\eta'$ does not leave its shine on γ_ℓ. To see this, we will prove by case analysis that one of the following four inequalities holds:

$$s_{\ell+1}(\eta) < s'_\ell[\tau'_\ell(\eta)], \quad s_{\ell+1}(\eta') < s'_\ell[\tau'_\ell(\eta')], \quad s_{\ell+1}(\eta) > s_{\ell+1}[\tau_{\ell+1}(\eta)], \quad \text{or} \quad s_{\ell+1}(\eta') > s_{\ell+1}[\tau_{\ell+1}(\eta')].$$

To see that the claim follows, note that if one of the first two inequalities holds, then $2D+\mathrm{div}(\eta+\eta')$ contains w_ℓ, hence $\eta+\eta'$ agrees with a departing function on γ_ℓ, a contradiction. If one of the second two inequalities holds, we see that $\eta+\eta'$ is shiny on $\gamma_{\ell+1}$, and hence does not leave its shine on γ_ℓ.

It therefore remains to show that one of the inequalities above holds. By Proposition 8.14, we may assume that the restriction of $D+\mathrm{div}(\eta)$ to $\gamma_\ell \smallsetminus \{v_\ell\}$ has degree 0. It follows that η agrees with φ on γ_ℓ, and $s_{\ell+1}(\eta) = s_{\ell+1}(\varphi)$. Since $\eta+\eta'$ is not departing on γ_ℓ, we have $s_{\ell+1}(\eta') = s_{\ell+1}(\varphi')-1$. By property ($\mathbf{B}'$), the bridge β_ℓ is not a switching bridge, so $\tau_{\ell+1}(\eta) = \tau'_\ell(\eta)$ and $\tau_{\ell+1}(\eta') = \tau'_\ell(\eta')$.

We now consider several cases. First, suppose that $s_{\ell+1}(\varphi) > s_{\ell+1}[\tau_{\ell+1}(\varphi)]$. If $\tau'_\ell(\eta) > \tau'_\ell(\varphi)$, then

$$s'_\ell[\tau'_\ell(\eta)] > s'_\ell[\tau'_\ell(\varphi)] \geq s_{\ell+1}(\varphi) = s_{\ell+1}(\eta).$$

On the other hand, if $\tau'_\ell(\eta) \leq \tau'_\ell(\varphi)$, then

$$s_{\ell+1}(\eta) = s_{\ell+1}(\varphi) > s_{\ell+1}[\tau_{\ell+1}(\varphi)] \geq s_{\ell+1}[\tau_{\ell+1}(\eta)].$$

Next, suppose that $s_{\ell+1}(\varphi') > s_{\ell+1}[\tau_{\ell+1}(\varphi')]$. If $\tau'_\ell(\eta') \geq \tau'_\ell(\varphi')$, then

$$s'_\ell[\tau'_\ell(\eta')] \geq s'_\ell[\tau'_\ell(\varphi')] \geq s_{\ell+1}(\varphi') > s_{\ell+1}(\eta').$$

On the other hand, if $\tau'_\ell(\eta') < \tau'_\ell(\varphi')$, then

$$s_{\ell+1}(\eta') = s_{\ell+1}(\varphi') - 1 > s_{\ell+1}[\tau_{\ell+1}(\varphi')] - 1 \geq s_{\ell+1}[\tau_{\ell+1}(\eta')]. \qquad \square$$

Proof of Theorem 8.21. If $\psi \in \mathcal{B}$ is not permissible on any loop, then $s_1(\psi) > 4$ or $s_{g'+1}(\psi) < 2$, and ψ achieves the minimum on the first or last bridge, respectively. By Lemma 8.25, if $\varphi + \varphi'$ and $\phi + \phi'$ are assigned to the first bridge and have the same slope, then the sets of slope indices $\{\tau'_0(\varphi), \tau'_0(\varphi')\}$ and $\{\tau'_0(\phi), \tau'_0(\phi')\}$ are the same.

We argue that every permissible function on the first block is assigned to a loop or bridge in the first block, or the bridge following the first block. The other blocks follow by a similar argument. We first consider the case where every non-skippable loop in the first block has at least one assigned function. For each loop γ_k in the block, we consider the number of cohorts on γ_k with the property that some function ψ in the cohort is not assigned to γ_k. We will show by induction that, for $k < z$, the number of such cohorts on γ_k is at most 2. We first show that there are at most three cohorts on γ_1, and at most two if γ_1 is skippable. To see this, it suffices to show that there are at most two pairs (i, j), with $i \leq j$, such that $s_1[i] + s_1[j] = 4$. Recall that, by assumption, we have either $s_1[5] \leq 2$ or $s_1[4] + s_1[6] \leq 5$. If $s_1[5] \leq 2$, then $s_1[i] + s_1[j] < 4$ for all $i < j \leq 5$. It follows that if $s_1[i] + s_1[j] = 4$, then either $i = j = 5$, or $j = 6$ and i is uniquely determined. On the other hand, if $s_1[4] + s_1[6] \leq 5$, then $s_1[i] + s_1[j] \leq 4$ for all pairs $i < j \leq 5$, with equality only if $i = 4$, $j = 5$. It follows that if $s_1[i] + s_1[j] = 4$, then either $i = 4$, $j = 5$, or $j = 6$ and i is uniquely determined.

If γ_1 is not skippable, then there is some $\psi \in \mathcal{B}$ that is assigned to γ_1. This function leaves its shine on γ_1, so by Proposition 8.35, any function in the same cohort is also assigned to γ_1. It follows that there are at most 2 cohorts such that some function ψ in the cohort is not assigned to γ_1.

As we proceed from left to right across the block, every time we reach a new loop, there are two possibilities. One possibility is that no function leaves its shine on γ_k, in which case by definition there are no more cohorts on γ_k than there are on γ_{k-1}. The other possibility is that some function ψ leaves its shine on γ_k. In this case, by assumption, there are permissible functions on γ_k, and γ_k is not skippable, so some function ψ' is assigned to γ_k. By Proposition 8.35, any function in the same cohort as ψ' is also assigned to γ_k. It follows that the number of cohorts on γ_k such that some function in the cohort is not assigned to γ_k is equal to the number of cohorts on γ_{k-1} such that some function in the cohort is not assigned to γ_{k-1}. Specifically, as we proceed from γ_{k-1} to γ_k, we introduce the cohort of ψ, but we remove the cohort of ψ'. By induction, therefore, the number of cohorts on γ_k with the property that some function in the cohort is not assigned to γ_k is at most 2.

By Proposition 8.29, no function is shiny on γ_z, and γ_z is not skippable. Combining this with our enumeration of cohorts in the preceding paragraph, we see that there are at most 2 cohorts on γ_z. By assumption, there is a function $\psi \in \mathcal{B}$ that is assigned to γ_z, and by Proposition 8.35, any function in the same cohort on γ_z is also assigned to γ_z. After assigning this cohort, there is at most one cohort left. Also by Lemma 8.36, if $\varphi + \varphi'$ and $\phi + \phi'$ are in the remaining cohort, then the sets of slope indices $\{\tau'_z(\varphi), \tau'_z(\varphi')\}$ and $\{\tau'_z(\phi), \tau'_z(\phi')\}$ are the same, hence $\varphi + \varphi'$ and $\phi + \phi'$ agree on γ_z. It follows that everything in the remaining cohort is assigned to the bridge β_{z+1}.

On the other hand, suppose that there is a non-skippable loop with no assigned function, and let γ_k be the first such loop. If ψ is a function that was permissible on an earlier loop but not permissible on γ_k, then there is a $k' < k$ such that ψ is a departing permissible function on $\gamma_{k'}$. By construction, this function must be assigned to loop $\gamma_{k'}$, or an earlier loop. It follows that all functions ψ that are permissible on loops $\gamma_{k'}$ for $k' < k$ have been assigned. Now, for each non-skippable loop $\gamma_{k'}$ with $k' > k$, there is at most one equivalence class of new permissible function on $\gamma_{k'}$, and on skippable loops, there are none. By construction, since there is only one equivalence class of unassigned permissible functions on $\gamma_{k'}$, this equivalence class is assigned to the loop $\gamma_{k'}$. In this way, every function that is permissible on the block is assigned to some loop. □

Theorem 8.21 shows that, if two functions are assigned to the same loop γ_k or the following bridge β_{k+1}, then they agree on γ_k. In fact, slightly more is true.

Lemma 8.37. *Suppose that both $\varphi + \varphi'$ and $\phi + \phi'$ are assigned to γ_k, or both are assigned to β_{k+1}. Then, after possibly reordering φ and φ', we have that φ agrees with ϕ on γ_k and φ' agrees with ϕ'*

on γ_k. Moreover, we have

$$\tau'_k(\varphi) = \tau'_k(\phi) \text{ and } \tau'_k(\varphi') = \tau'_k(\phi').$$

Proof. In the case where the two functions are assigned to the same loop, this is simply Lemma 8.32. It therefore suffices to consider the case where they are assigned to the same bridge. We see from the proof of Theorem 8.21 that, if $\varphi + \varphi'$ and $\phi + \phi'$ are assigned to the bridge β_{k+1}, then they are in the same cohort on γ_k. By Proposition 8.35, if the set of slope indices $\{\tau'_k(\varphi), \tau'_k(\varphi')\}$ is different from $\{\tau'_k(\varphi), \tau'_k(\varphi')\}$, then one of the two functions is assigned to the loop γ_k. It follows that, if both functions are assigned to the bridge β_{k+1}, then these two sets of slope indices are the same, and the conclusion holds by property (**A**). □

9. Constructing the tropical independence

Recall that in §8 we considered a linear series V of rank 6 on the curve X with certain imposed ramification conditions, and used this to construct the master template θ as a tropical linear combination of pairwise sums of building block functions on the skeleton Γ. Our goal in this section is to construct an independence from a set $\mathcal{T}$ of 28 pairwise sums of functions in $\Sigma = \text{trop}(V)$. This independence, denoted $\vartheta_{\mathcal{T}}$, will be the best approximation of θ by a tropical linear combination of the functions in $\mathcal{T}$; see Definition 9.2.

We retain the standing assumptions that $g = 22$ or 23, $g' \in \{g, g-1, g-2\}$, and Γ is a chain of g' loops with admissible edge lengths, as shown in Figure 4.

Theorem 9.1. *Let X be a curve of genus g' with skeleton Γ, and let $p \in X$ specialize to w_0. Let V be a linear series of degree $g+3$ and rank 6 on X, and let $\Sigma = \text{trop}\, V$. Assume that*

(i) *if $g' = g-1$ then $a_1^V(p) \geq 2$, and*
(ii) *if $g' = g-2$ then either $a_1^V(p) \geq 3$ or $a_0^V(p) + a_2^V(p) \geq 5$.*

Then there is an independence ϑ among 28 pairwise sums of functions in Σ.

When $g' = g$, this proves Theorem 1.3. When g' is equal to $g-1$ or $g-2$, we have analogous consequences for multiplication maps for linear series with ramification on a general pointed curve of genus g'. See Theorems 10.1 and 10.2, respectively. All three are used in the proof of Theorem 1.4.

We prove Theorem 9.1 by considering cases depending on the number of switching loops and bridges, and sub-cases depending on the decreasing bridges, or the relationship between the switching loops, when there are two. Our basic strategy is the same in each case. First, we identify a collection $\mathcal{S}$ of 7 or more functions in Σ, each of which is either a building block or has a relatively simple expression as a tropical linear combination of building blocks. When there are more than 7 functions in $\mathcal{S}$, then $\mathcal{S}$ is necessarily tropically dependent. Next, taking into consideration the combinatorial properties of the dependences among the functions in $\mathcal{S}$, we specify a set $\mathcal{A}$ of building blocks satisfying property (**A**), and a set $\mathcal{B}$ of pairwise sums of elements of $\mathcal{A}$ satisfying (**B**) and (**B′**). Having specified $\mathcal{A}$ and $\mathcal{B}$, we run the algorithm from §8 to construct a master template θ. We then specify a collection $\mathcal{T}$ of 28 pairwise sums of functions in $\mathcal{S}$, and consider the best approximation of the template θ by $\mathcal{T}$, defined as follows.

Definition 9.2. Let $\mathcal{T}$ be a finite subset of PL(Γ). The *best approximation* of $\theta \in \text{PL}(\Gamma)$ by $\mathcal{T}$ is

$$\vartheta_{\mathcal{T}} := \min\{\varphi - c(\varphi, \theta) : \varphi \in \mathcal{T}\}, \tag{31}$$

where $c(\varphi, \theta) = \min\{\varphi(v) - \theta(v) : v \in \Gamma\}$.

Note that $\vartheta_{\mathcal{T}} \geq \theta$. Moreover, for each $\varphi \in \mathcal{T}$ there is some point $v \in \Gamma$ where φ achieves the minimum in the definition of $\vartheta_{\mathcal{T}}$ and $\vartheta_{\mathcal{T}}(v) = \theta(v)$. Hence $\vartheta_{\mathcal{T}}$ is indeed the best approximation of θ from above, among all tropical linear combinations of functions in $\mathcal{T}$. We prove each case of Theorem 9.1 by showing that this best approximation $\vartheta_{\mathcal{T}}$ is an independence.

We will often consider the best approximation of θ by a single function φ, and then we talk about the subset of Γ where this approximation *achieves equality*, i.e., $\{v \in \Gamma : \varphi(v) - c(\varphi, \theta) = \theta(v)\}$. We will repeatedly use the following lemma, which tells us about the locus where the best approximation of θ by φ achieves equality, in the cases where φ is a tropical linear combination of functions in $\mathcal{B}$.

Lemma 9.3. *Let $\theta = \min_{\psi \in \mathcal{B}}\{\psi + a_\psi\}$. Suppose $\varphi = \min_{\psi' \in \mathcal{B}'}\{\psi' + b_{\psi'}\}$, where $\mathcal{B}' \subset \mathcal{B}$. Then the best approximation of θ by φ achieves equality on the entire region where some $\psi' \in \mathcal{B}'$ achieves the minimum in θ.*

Proof. Let $c = \min_{\psi' \in \mathcal{B}'}\{b_{\psi'} - a_{\psi'}\}$. Choose $\psi' \in \mathcal{B}'$ such that $c = b_{\psi'} - a_{\psi'}$. Then $\varphi - c \geq \theta$, with equality at points where ψ' achieves the minimum in θ. □

We now outline the cases to be considered in our proof of Theorem 9.1. Recall that switching loops have positive multiplicity, switching bridges have multiplicity at least 2, and the sum of all multiplicities is at most 2. Therefore, Σ falls into one of the following cases:

(1) There are no switching loops or bridges.
(2) There is a switching bridge.
(3) There is one switching loop.
(4) There are two switching loops.

The case of two switching loops is the most delicate, and we consider subcases depending on the relationship between the two switching loops.

9.1. **Case 1: no switching loops or bridges.** Suppose there are no switching loops or bridges. By Corollary 6.23, for $0 \leq i \leq 6$, there is a function $\varphi_i \in \Sigma$ such that

$$s_k(\varphi_i) = s_k[i] \text{ and } s'_k(\varphi_i) = s'_k[i], \text{ for all } k.$$

We set $\mathcal{S} = \{\varphi_i : 0 \leq i \leq 6\}$, and show that $\mathcal{T} = \{\varphi_i + \varphi_j : 0 \leq i, j \leq 6\}$ is tropically independent.

9.1.1. *Case 1a: no decreasing bridges.* If there are no decreasing bridges, then each φ_i has constant slope along each bridge, and the slope index sequence associated to φ_i is the constant sequence i. It follows that φ_i is a building block. Let $\varphi_{ij} = \varphi_i + \varphi_j$. If $i \neq j$, then $\tau'_k(\varphi_i) \neq \tau'_k(\varphi_j)$ for any k, and thus property (**A**) is satisfied. Properties (**B**) and (**B′**) are satisfied vacuously, and hence we may run the algorithm in §8 to construct the master template. By Lemma 8.37, only one function is assigned to each loop and to each of the bridges between the blocks. It follows that the master template θ is itself an independence in this case. This proves Theorem 9.1 in Case 1a. □

9.1.2. *Case 1b: one decreasing bridge, of multiplicity one.* In this case, there is one index h and one bridge β_ℓ such that the slope of φ_h decreases on β_ℓ, and it decreases by exactly 1.

Note that φ_h can be written as a tropical linear combination of two building blocks, both of which agree with φ_h on every loop, and have the same slope on every bridge other than β_ℓ, but with slopes $s_\ell[h]$ and $s_\ell[h] + 1$ on β_ℓ. We label them φ_h^0 and φ_h^∞, respectively.

Lemma 9.4. *In this case, the set $\mathcal{A} := \{\varphi_i : i \neq h\} \cup \{\varphi_h^0, \varphi_h^\infty\}$ satisfies property* (**A**).

Proof. Every function in $\mathcal{A}$ has constant slope along each bridge. By construction, the slope index sequence associated to φ_i is the constant sequence i, and the slope index sequence associated to both φ_h^0 and φ_h^∞ is the constant sequence h. In particular, each function in $\mathcal{A}$ is a building block. Moreover, since φ_h^0 and φ_h^∞ agree on every loop, $\mathcal{A}$ satisfies property (**A**). □

Lemma 9.5. *The set $\mathcal{B}$ of pairwise sums of elements of $\mathcal{A}$ satisfies properties* (**B**) *and* (**B′**).

Proof. Recall that $\mathcal{B}$ satisfies property (**B**) if, whenever there is a permissible function $\varphi + \varphi' \in \mathcal{B}$ on γ_k such that $2D + \mathrm{div}(\varphi + \varphi')$ contains w_k, and either γ_k switches slope $\tau_k(\varphi)$ or $s_{k+1}(\varphi) < s'_k[\tau'_k(\varphi)]$, then there is some permissible function $\psi \in \mathcal{B}$ that agrees with $\varphi + \varphi'$ on $\Gamma_{\leq k}$ such that $s_{k+1}(\psi) > s_{k+1}(\varphi + \varphi')$. In this case, note that φ_h^0 is the only function in $\mathcal{A}$ satisfying $s_k(\varphi_h^0) < s'_{k-1}[\tau'_k(\varphi_h^0)]$, and then only when $k = \ell$. Because φ_h^∞ agrees with φ_h^0 to the left of β_ℓ and has larger slope on β_ℓ, we see that $\mathcal{B}$ satisfies property (**B**). Property (**B′**) is satisfied vacuously. □

Because the set $\mathcal{B}$ satisfies properties (**B**) and (**B′**), we may construct the master template θ.

Proof of Theorem 9.1, Case 1b. By Lemma 8.37, if two functions $\psi, \psi' \in \mathcal{B}$ are assigned to the same loop or bridge, then these two functions must be either

$$\psi = \varphi_h^0 + \varphi, \psi' = \varphi_h^\infty + \varphi \text{ for some } \varphi \in \mathcal{A}, \text{ or}$$

$$\psi = \varphi_h^0 + \varphi_h^0, \psi' = \varphi_h^\infty + \varphi_h^\infty.$$

Thus, in the master template θ, for each function φ_{ij} with $i, j \neq h$, there is a point where φ_{ij} achieves the minimum uniquely in both θ and $\vartheta_{\mathcal{T}}$.

By Lemma 9.3, the region where φ_{hj} achieves equality in $\vartheta_{\mathcal{T}}$ contains the region where one of $\varphi_h^0 + \varphi_j, \varphi_h^\infty + \varphi_j$ achieves the minimum in the master template θ, and it does so uniquely. Similarly, the region where φ_{hh} achieves equality in $\vartheta_{\mathcal{T}}$ contains the region where one of $\varphi_h^0 + \varphi_h^0, \varphi_h^0 + \varphi_h^\infty, \varphi_h^\infty + \varphi_h^\infty$ achieves the minimum in the master template θ, and it does so uniquely. We conclude that $\vartheta_{\mathcal{T}}$ is an independence, as required. □

9.1.3. *Case 1c: remaining cases without switching loops or bridges.* The remaining possibilities are that there may be two decreasing bridges of multiplicity 1, or one decreasing bridge of multiplicity 2. We choose the set $\mathcal{A}$ in a similar way to the previous case, now containing up to 9 functions. Specifically, we define a function $\varphi \in R(D)$ to be in $\mathcal{A}$ if there is an i such that:

(i) on every loop γ_k, φ agrees with φ_i, and
(ii) on every bridge β_k, there is a tangent vector ζ such that φ has constant slope $s_\zeta(\varphi_i)$ on β_k.

The proofs that $\mathcal{A}$ satisfies property (**A**) and that the set $\mathcal{B}$ consisting of all pairwise sums of functions in $\mathcal{A}$ satisfies (**B**) and (**B′**) are similar to Lemmas 9.4 and 9.5. The remainder of the argument, showing that the best approximation $\vartheta_{\mathcal{T}}$ to θ is an independence, is similar to Case 1b.

9.2. **Case 2: a switching bridge.** Suppose β_ℓ is a switching bridge. As discussed in §6.8, there is a unique index h such that β_ℓ switches slope h. Moreover, β_ℓ has multiplicity 2 and

$$s'_{\ell-1}[h+1] = s'_{\ell-1}[h] + 1 = s_\ell[h+1] + 1 = s_\ell[h] + 2. \tag{32}$$

By Lemma 6.22, for $j \notin \{h, h+1\}$, there is $\varphi_j \in \Sigma$ with $s_k(\varphi_j) = s_k[j]$ and $s'_k(\varphi_j) = s'_k[j]$ for all k. These functions φ_j are building blocks.

Lemma 9.6. *There is a unique point $x \in \beta_\ell$ where the incoming and outgoing slopes, denoted s_x and s'_x, resepctively, satisfy $s_x[i] = s'_{\ell-1}[i]$ and $s'_x[i] = s_\ell[i]$ for all i.*

Proof. The argument is similar to Case 2 of Example 6.8. □

We now identify a subset $\mathcal{S} \subset \Sigma$. It will consist of the functions φ_j for $j \notin \{h, h+1\}$, plus three more functions that are contained in the tropicalization of a pencil and characterized in Proposition 9.7. They are closely analogous to the functions ψ_A, ψ_B and ψ_C in Example 6.8.

Proposition 9.7. *There is a pencil $W \subset V$ and functions φ_A, φ_B, and φ_C in* trop(W) *with the following properties:*

(i) $s'_k(\varphi_A) = s'_k[h]$ *for all* $k < \ell$, *and* $s_x(\varphi_A) = s_x[h]$;
(ii) $s'_x(\varphi_B) = s'_x[h+1]$, *and* $s_k(\varphi_B) = s_k[h+1]$ *for all* $k \geq \ell$;
(iii) $s'_k(\varphi_C) = s'_k[h+1]$ *for all* $k < \ell$, *and* $s_k(\varphi_C) = s_k[h]$ *for all* $k \geq \ell$;
(iv) $s_k(\varphi_\bullet) \in \{s_k[h], s_k[h+1]\}$ *and* $s'_k(\varphi_\bullet) \in \{s'_k[h], s'_k[h+1]\}$ *for all* k.

We find it helpful to illustrate the essential properties of these functions in Figure 14, which provides a "zoomed out" view in which the chain of loops looks like an interval. A region labeled h in this interval indicates that φ has slope $s_k(\varphi) = s_k[h]$ and $s'_k(\varphi) = s'_k[h]$ for all k in the given region. We include similar schematic illustrations in all subsequent cases.

Proof. By Lemma 6.4, there is $\varphi_A \in \Sigma$ such that $s'_0(\varphi_A) \leq s'_0[h]$, and $s_{g'+1}(\varphi_A) \geq s_{g'+1}[h]$. Since β_ℓ is the only switching bridge, and there are no switching loops, we have $s'_k(\varphi_A) \leq s'_k[h]$ for $k < \ell$, and $s_k(\varphi_A) \geq s_k[h]$ for $k \geq \ell$. In particular, $s_\ell(\varphi_A) \geq s_\ell[h]$, so $s_x(\varphi_A) \geq s_x[h]$, and it follows that $s_x(\varphi_A) = s_x[h]$. This proves (i), because there are no switching loops or bridges to the left of β_ℓ. The proof of (ii) is similar.

We now prove (iii). Given φ_A and φ_B in Σ satisfying (i) and (ii), choose f_A and $f_B \in V$ tropicalizing to φ_A and φ_B, respectively. Let W be the pencil spanned by f_A and f_B. Arguments similar to the proof of (i) above show that $s_k(\mathrm{trop}(W)) = (s_k[h], s_k[h+1])$, for all k, and $s_x(\mathrm{trop}(W)) = (s_x[h], s_x[h+1])$. Choose a function $f \in W$ such that $\varphi = \mathrm{trop}(f)$ satisfies $s_x(\varphi) = s_x[h+1]$. Then $s'_k(\varphi) = s'_k[h+1]$ for $k < \ell$. Similarly, choose $\varphi' \in \mathrm{trop}(W)$ such that $s'_x(\varphi') = s'_x[h]$, and $s_k(\varphi') = s_k[h]$ for $k > \ell$. Finally, by adding a scalar to φ', we may assume that $\varphi(x) = \varphi'(x)$ and set $\varphi_C = \min\{\varphi, \varphi'\}$. □

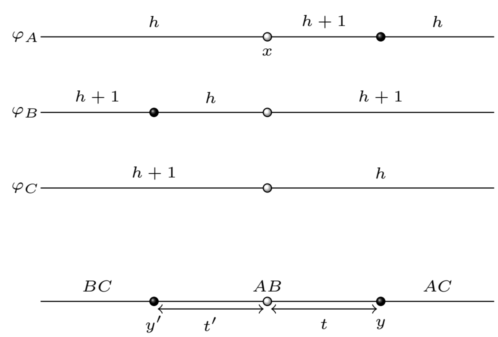

FIGURE 14. The first three lines give a schematic depiction of the three functions φ_A, φ_B, φ_C from Proposition 9.7. The bottom line illustrates the tropical dependence among them, and is analogous to the bottom line in Figure 3.

Having fixed $\mathcal{S} = \{\varphi_j : j \neq h, h+1\} \cup \{\varphi_A, \varphi_B, \varphi_C\}$, we now describe the set of building blocks $\mathcal{A}$. It will include $\{\varphi_j : j \neq h, h+1\}$ along with three additional functions, as follows.

Lemma 9.8. *There are building blocks $\varphi^0_h, \varphi^0_{h+1}$, and φ^∞_h in $R(D)$ such that*

(i) $s_k(\varphi^0_h) = s_k[h]$ *for all k;*
(ii) $s_k(\varphi^0_{h+1}) = s'_{k-1}[h+1]$ *for all k;*
(iii) $s_k(\varphi^\infty_h) = s_k[h]$ *for all $k < \ell$, and $s_k(\varphi^\infty_h) = s_k[h+1]$ for all $k \geq \ell$.*

Proof. To construct φ^∞_h, consider a function that agrees with φ_A to the left of x and with φ_B to the right of x. Because these two functions have the same slope at x, they "glue" together to give a function in $R(D)$. The construction of the other two functions is similar. □

Lemma 9.9. *The set $\mathcal{A} := \{\varphi_i : i \neq h, h+1\} \cup \{\varphi^0_h, \varphi^0_{h+1}, \varphi^\infty_h\}$ satisfies property* (**A**).

Proof. The slope index sequences of φ^0_h and φ^0_{h+1} are the constant sequences h and $h+1$, respectively. We then have $\tau_k(\varphi^\infty_h) = \tau'_k(\varphi^\infty_h) = h$ for $k < \ell$, and $\tau_k(\varphi^\infty_h) = \tau'_k(\varphi^\infty_h) = h+1$ for $k \geq \ell$. Thus, if two different functions $\varphi, \varphi' \in \mathcal{A}$ satisfy $\tau'_k(\varphi) = \tau'_k(\varphi')$, then without loss of generality $\varphi = \varphi^\infty_h$. Moreover, either $\varphi' = \varphi^0_h$ and $k < \ell$, or $\varphi' = \varphi^0_{h+1}$ and $k \geq \ell$. In either case, we see that φ agrees with φ' on γ_k, so $\mathcal{A}$ satisfies property (**A**). □

Note that the slope of the function φ^0_h along β_ℓ is $s_\ell[h]$, which is not in $s'_{\ell-1}(\Sigma)$. Hence φ^0_h cannot be in Σ. Similarly, the function φ^0_{h+1} cannot be in Σ. However, the functions φ_A, φ_B, and φ_C can be written as tropical linear combinations of $\varphi^0_h, \varphi^0_{h+1}$, and φ^∞_h, as follows.

Lemma 9.10. *The functions φ_A, φ_B, and φ_C can be written as tropical linear combinations of the building blocks $\varphi_h^0, \varphi_{h+1}^0$, and φ_h^∞, as follows:*

(i) *The function $\varphi_A \in \Sigma$ is a tropical linear combination of the functions φ_h^0 and φ_h^∞, where the two functions simultaneously achieve the minimum at a point y to the right of x.*
(ii) *The function φ_B is a tropical linear combination of the functions φ_{h+1}^0 and φ_h^∞, where the two functions simultaneously achieve the minimum at a point y' to the left of x.*
(iii) *The function φ_C is a tropical linear combination of the functions φ_h^0 and φ_{h+1}^0, where the two functions simultaneously achieve the minimum at x.*

Proof. This is very similar to Example 6.8. By definition, φ_A has slope $s_k[h]$ on all bridges β_k to the left of β_ℓ. Because β_ℓ is switches slope h, φ_A can have slope $s_k[h+1]$ for some distance to the right of x, at which point its slope drops to $s_k[h]$ and continues to be $s_k[h]$ for the rest of the way. To the left of this point, φ_A looks like φ_h^∞, and to the right, it looks like φ_h^0. The functions φ_B and φ_C can be described similarly. □

Definition 9.11. Let t be the distance, measured along the bridges and bottom edges, from x to y. Similarly, let t' be the distance, measured along the bridges and bottom edges, from x to y'.

As in Example 6.8, the functions φ_A, φ_B and φ_C are tropically dependent, and this induces a relation between the parameters t and t'.

Proposition 9.12. *The distance t' is an increasing piecewise affine function in t.*

Proof. If we consider the point y at which the function φ_A equals φ_h^∞ to the left and equals φ_h^0 to the right, we see that locally in a neighborhood of this point, φ_B agrees with φ_h^∞ and φ_C agrees with φ_h^0. Thus, in the tropical dependence between these three functions, all three must achieve the minimum at this point. This determines the other point, to the left of x, where all three achieve the minimum, which by the same reasoning is y'. The condition that all three functions are equal at these two points yields a system of equations, and by solving for t', we obtain an expression for t' as an increasing piecewise affine function in t. □

Note that φ_A is linear with slope $s'_{\ell-1}[h] = s_\ell[h+1]$ on a subinterval of β_ℓ. This subinterval extends from the left endpoint $w_{\ell-1}$ of β_ℓ to the point to the right of x of distance $\min\{t, d(x, v_\ell)\}$.

Definition 9.13. Let $I \subset \beta_\ell$ be the interval where φ_A has slope $s'_{\ell-1}[h]$.

Corollary 9.14. *If I has length less than $m_{\ell-1}$, then $t' = t$.*

Proof. If y is not contained in the bridge β_ℓ, then φ_A has slope $s'_{\ell-1}[h]$ on the entire bridge β_ℓ. The assumption therefore implies that y is contained in the bridge β_ℓ. The point y' is contained either in the bridge β_ℓ or the bottom edge of the loop $\gamma_{\ell-1}$. We consider the case where y' is contained in the bridge first. Examining the tropical dependence described in Proposition 9.12, we see that

$$\left(s'_{\ell-1}[h+1] - s'_{\ell-1}[h]\right) t' = \left(s_\ell[h+1] - s_\ell[h]\right) t.$$

But, by equation 32, we have $s'_{\ell-1}[h+1] - s'_{\ell-1}[h] = s_\ell[h+1] - s_\ell[h] = 1$, and the result follows.

We now consider the case where y' is contained in the bottom edge of the loop $\gamma_{\ell-1}$. Recall that $\mu(\gamma_{\ell-1}) = 0$. It follows that φ_{h+1}^0 has slope one greater than φ_h^0 along this bottom edge. The result then follows by the same argument as the previous case. □

Within this case, there are special subcases where one must sometimes choose $\mathcal{B}$ to be a proper subset of the set of all pairwise sums of functions in $\mathcal{A}$, in order to ensure it satisfies property (**B′**).

Definition 9.15. Let $\widetilde{\mathcal{B}}$ be the set of pairwise sums of elements of $\mathcal{A}$. Suppose $s_{\ell-1}[h] < s'_{\ell-1}[h]$, and there are functions $\varphi, \varphi' \in \mathcal{A}$ such that $s_{\ell-1}(\theta) = s_\ell[h] + s_\ell(\varphi) = s_\ell[h] + s_\ell(\varphi') + 1$. Then

(i) if I has length less than $m_{\ell-1}$, let $\mathcal{B} = \widetilde{\mathcal{B}} \smallsetminus \{\varphi_h^\infty + \varphi'\}$;
(ii) if I has length at least $m_{\ell-1}$, let $\mathcal{B} = \widetilde{\mathcal{B}} \smallsetminus \{\varphi_h^0 + \varphi\}$.

Otherwise, let $\mathcal{B} = \widetilde{\mathcal{B}}$.

Lemma 9.16. *This set $\mathcal{B}$ satisfies properties* (**B**) *and* (**B′**).

Proof. As in Lemma 9.5, φ_h^0 is the only function in $\mathcal{A}$ satisfying $s_k(\varphi_h^0) < s'_{k-1}[\tau'_k(\varphi_h^0)]$, and then only when $k = \ell$. If $\varphi_h^0 + \varphi \in \mathcal{B}$ is permissible on $\gamma_{\ell-1}$ for some $\varphi \in \mathcal{A}$, then by Definition 9.15 we see that $\varphi_h^\infty + \varphi$ is also in $\mathcal{B}$. But $\varphi_h^0 + \varphi$ agrees with $\varphi_h^\infty + \varphi$ to the left of β_ℓ, and the latter function has higher slope along β_ℓ, so $\mathcal{B}$ satisfies property (**B**).

To establish property (**B′**), we must show that if two permissible functions in $\mathcal{B}$ agree on $\gamma_{\ell-1}$ and have different slopes on β_ℓ, then no function is shiny on $\gamma_{\ell-1}$. We first show that every shiny function on $\gamma_{\ell-1}$ is in fact new. This follows from Example 8.7. Specifically, since both $\beta_{\ell-1}$ and $\gamma_{\ell-1}$ have multiplicity zero, the restriction of $D + \mathrm{div}(\varphi)$ to $\gamma_{\ell-1} \smallsetminus \{w_{\ell-1}\}$ has degree at most 1 for every function $\varphi \in \mathcal{A}$. Moreover, none of these divisors contain $v_{\ell-1}$ in their support, which implies that every shiny function on $\gamma_{\ell-1}$ is new.

Now, if two functions in $\mathcal{B}$ are permissible on $\gamma_{\ell-1}$ and have different slopes on β_ℓ, then the one with higher slope must be departing. In addition, if the two functions agree on $\gamma_{\ell-1}$, then they must be $\varphi_h^0 + \varphi$ and $\varphi_h^\infty + \varphi$ for some $\varphi \in \mathcal{A}$. We may therefore assume that $\varphi_h^\infty + \varphi$ is departing on $\gamma_{\ell-1}$ and

$$s_{\ell-1}(\theta) = s_\ell[h] + s_\ell(\varphi). \tag{33}$$

Since $\varphi_h^\infty + \varphi$ is departing, either (i) $s_{\ell-1}(\varphi_h^\infty) < s_\ell(\varphi_h^\infty)$, or (ii) $s_{\ell-1}(\varphi) < s_\ell(\varphi)$ and $\varphi \neq \varphi_h^\infty$.

If $s_{\ell-1}(\varphi_h^\infty) < s_\ell(\varphi_h^\infty)$, then any new function must be of the form $\varphi_h^\infty + \varphi'$, where

$$s_{\ell-1}(\theta) = s_\ell[h] + s_\ell(\varphi') + 1.$$

Definition 9.15 ensures that either $\varphi_h^\infty + \varphi'$ is not in $\mathcal{B}$, in which case no function is shiny on $\gamma_{\ell-1}$, or $\varphi_h^0 + \varphi$ is not in $\mathcal{B}$, in which case no two permissible functions agree on $\gamma_{\ell-1}$.

It remains to consider the case where $s_{\ell-1}(\varphi) < s_\ell(\varphi)$, and $\varphi \neq \varphi_h^\infty$. Then any new function must be of the form $\varphi + \varphi'$, where

$$s_{\ell-1}(\theta) = s_\ell(\varphi) + s_\ell(\varphi').$$

Combining this with (33), we see that $s_\ell(\varphi') = s_\ell[h]$. The only function in $\mathcal{A}$ with this slope is φ_h^0, so $\varphi' = \varphi_h^0$. Since $\varphi \neq \varphi_h^\infty$, we have $s_{\ell-1}(\varphi_h^\infty) \geq s_\ell(\varphi_h^\infty)$. It follows that $s_{\ell-1}(\varphi_h^0) > s_\ell(\varphi_h^0)$. Hence the function $\varphi + \varphi_h^0$ is not new, and no function is shiny on $\gamma_{\ell-1}$. □

Since $\mathcal{A}$ and $\mathcal{B}$ satisfy properties (**A**), (**B**), and (**B′**), we may run the algorithm from Section 8 to construct the master template θ. The next step in our argument is to describe the set $\mathcal{T}$ of 28 pairwise sums of functions in $\mathcal{S} = \{\varphi_j : j \neq h, h+1\} \cup \{\varphi_A, \varphi_B, \varphi_C\}$ from which we will construct an independence. If $i, j \notin \{h, h+1\}$, then $\varphi_{ij} \in \mathcal{T}$. The remaining functions may be thought of as replacements for the functions in $\mathcal{B}$ that contain φ_h^0, φ_{h+1}^0, or φ_h^∞ as a summand.

For $j \notin \{h, h+1\}$, we denote $\varphi_{h,j}^0 = \varphi_h^0 + \varphi_j$, and similarly for $\varphi_{h+1,j}^0$ and $\varphi_{h,j}^\infty$. We will replace $\mathcal{B}_j = \mathcal{B} \cap \{\varphi_{h,j}^0, \varphi_{h+1,j}^0, \varphi_{h,j}^\infty\}$ with either $\{\varphi_A + \varphi_j, \varphi_C + \varphi_j\}$ or $\{\varphi_B + \varphi_j, \varphi_C + \varphi_j\}$, depending on where the best approximation of θ by $\varphi_C + \varphi_j$ achieves equality, as follows.

Lemma 9.17. *The best approximation of θ by $\varphi_C + \varphi_j$ achieves equality on the region where either $\varphi_{h,j}^0$ or $\varphi_{h+1,j}^0$ achieves the minimum.*

Proof. If $\mathcal{B}$ contains both $\varphi_{h,j}^0$ and $\varphi_{h+1,j}^0$ then this is immediate from Lemmas 9.3 and 9.10(iii). Otherwise, we are in the subcase of Definition 9.15(ii) where $\varphi = \varphi_j$. Then Lemma 9.3 does not apply, since $\varphi_C + \varphi_j$ is not a tropical combination of functions in $\mathcal{B}$. In this case, $\varphi_C + \varphi_j$ has slope greater than $s_{\ell-1}(\theta)$ on β_ℓ, and so the best approximation cannot achieve equality to the right of $\gamma_{\ell-1}$. Hence it must achieve equality to the left of $\gamma_{\ell-1}$, where $\varphi_C + \varphi_j$ agrees with $\varphi_{h+1,j}^0$. □

We note that, a priori, it is possible for this best approximation to achieve equality on *both* regions. However, in our construction of the master template θ, if we perturb the coefficients of all functions in $\mathcal{B}$ that are assigned to the same loop or bridge by some small value ϵ, this does not change the conclusion of Theorem 8.21. We may therefore assume that it achieves equality on exactly one of these two regions. If the best approximation by $\varphi_C + \varphi_j$ achieves equality where $\varphi^0_{h,j}$ achieves the minimum, then we replace $\mathcal{B}_j$ with $\{\varphi_B+\varphi_j, \varphi_C+\varphi_j\}$. Otherwise, it achieves equality where $\varphi^0_{h+1,j}$ achieves the minimum, and we replace $\mathcal{B}_j$ with $\{\varphi_A + \varphi_j, \varphi_C + \varphi_j\}$.

Similarly, we replace the subset of $\mathcal{B}$ consisting of pairwise sums of elements of $\{\varphi^0_h, \varphi^0_{h+1}, \varphi^\infty_h\}$ with three pairwise sums of elements of $\{\varphi_A, \varphi_B, \varphi_C\}$. In all cases, we put $\varphi_C + \varphi_C$ in $\mathcal{T}$. If the best approximation of θ by $\varphi_C + \varphi_C$ achieves equality on a region to the left of x, then we put $\varphi_A + \varphi_C$ in $\mathcal{T}$. Otherwise, we put $\varphi_B + \varphi_C$ in $\mathcal{T}$. If $\varphi_A + \varphi_C \in \mathcal{T}$ and the best approximation of θ by $\varphi_A + \varphi_C$ achieves equality on a region to the left of x, then we put $\varphi_A + \varphi_A$ in $\mathcal{T}$. If it achieves equality on a region to the right of x, then we put $\varphi_A + \varphi_B$ in $\mathcal{T}$. Similarly, if $\varphi_B + \varphi_C \in \mathcal{T}$ and the best approximation of θ by $\varphi_B + \varphi_C$ achieves equality on a region to the left of x, then we put $\varphi_A+\varphi_B$ in $\mathcal{T}$. If it achieves equality on a region to the right of x, then we put $\varphi_B+\varphi_B$ in $\mathcal{T}$. These choices are made so that the three chosen pairwise sums of elements of $\{\varphi_A, \varphi_B, \varphi_C\}$ do not agree on the regions where they achieve the minimum. By an argument similar to that of Lemma 9.17, in the best approximation of θ by $\mathcal{T}$, each of these functions will achieve equality on the region where one of the building blocks in $\mathcal{B}$ achieves the minimum in θ.

Proof of Theorem 9.1, case 2. We first consider the case where the set $\mathcal{B}$ consists of all pairwise sums of elements of $\mathcal{A}$. By Lemmas 9.3 and 9.10, each of the 28 functions in $\mathcal{T}$ achieves the minimum on a region where one of the functions in $\mathcal{B}$ achieves the minimum in the master template θ. We show that each function achieves the minimun *uniquely* at some point of Γ. By Lemma 8.37, if two functions $\psi, \psi' \in \mathcal{B}$ are assigned to the same loop γ_{k-1} or bridge β_k, then $\psi = \varphi^\infty_h + \varphi$ for some $\varphi \in \mathcal{A}$, and either

$$\psi' = \varphi^0_h + \varphi, k \leq \ell, \text{ or } \psi' = \varphi^0_{h+1} + \varphi, k > \ell.$$

Assume for simplicity that $\varphi = \varphi_j$ for some j. The other cases are similar.

The best approximation of θ by $\varphi_C+\varphi_j$ achieves equality on the region where either $\varphi^0_{h,j}$ or $\varphi^0_{h+1,j}$ achieves the minimum in θ. Suppose it does so on the region where $\varphi^0_{h,j}$ achieves the minimum. (The other case is similar.) In this case, by construction, the set $\mathcal{T}$ does not contain $\varphi_A + \varphi_j$. Since $\varphi_C + \varphi_j$ does not agree with any other pairwise sum of functions in $\mathcal{S}$ on the loop or bridge where $\varphi^0_{h,j}$ is assigned, it must achieve the minimum uniquely. A similar argument shows that $\varphi_B + \varphi_j$ achieves the minimum uniquely on the bridge or loop where either $\varphi^0_{h+1,j}$ or $\varphi^\infty_{h,j}$ is assigned. This completes the proof that every function in $\mathcal{T}$ achieves the minimum uniquely, and hence $\vartheta_{\mathcal{T}}$ is an independence, in the cases where $\mathcal{B}$ contains all pairwise sums of elements of $\mathcal{A}$.

We now turn to the cases where some function is omitted from the set $\mathcal{B}$. In these cases it suffices to show that the best approximation of θ by functions in $\mathcal{T}$ achieves equality on a region where some pairwise sum of building blocks in $\mathcal{B}$ achieves the minimum. The conclusion will then follow from the argument of the previous two paragraphs. Fix functions φ and φ' as in Definition 9.15. Suppose that I has length greater than or equal to $m_{\ell-1}$. In this case Lemma 9.3 does not apply, since the functions $\varphi_A+\varphi$ and $\varphi_C+\varphi$ are not tropical linear combinations of functions in $\mathcal{B}$. By Lemma 9.17, however, the best approximation of θ by $\varphi_C + \varphi_j$ achieves equality on the region where $\varphi^0_{h+1} + \varphi_j$ achieves the minimum. By an identical argument, the best approximation of θ by $\varphi_A + \varphi_j$ achieves equality where $\varphi^\infty_h + \varphi_j$ achieves the minimum.

Now, suppose that I has length less than $m_{\ell-1}$, so $\mathcal{B} = \widetilde{\mathcal{B}} \smallsetminus \{\varphi^\infty_h + \varphi'\}$. We will consider the case where $\varphi_A + \varphi' \in \mathcal{T}$. The case where $\varphi_B + \varphi' \in \mathcal{T}$ is similar. Note that Lemma 9.3 does not apply, since the function $\varphi_A + \varphi'$ is not a tropical linear combination of functions in $\mathcal{B}$. The assumption that I has length less than $m_{\ell-1}$ implies that $\varphi_A+\varphi'$ has smaller slope than θ on a large subinterval of β_ℓ, and slope smaller than or equal to that of θ on every bridge to the left of β_ℓ. Thus, in the best approximation, $\varphi_A + \varphi'$ must obtain the minimum to the right of β_ℓ. The assumption on the length of of I also implies that $\varphi_A+\varphi'$ agrees with $\varphi^0_h+\varphi'$ to the right of β_ℓ, hence $\varphi_A+\varphi'$ achieves the minimum on the loop or bridge to which $\varphi^0_h + \varphi'$ is assigned. □

9.3. **Case 3: one switching loop.** We now consider the case where there is only one switching loop γ_ℓ, which switches slope h. By Lemma 6.22, for all $j \notin \{h, h+1\}$, there is $\varphi_j \in \Sigma$ with

$$s_k(\varphi_j) = s_k[j] \text{ and } s'_k(\varphi_j) = s'_k[j] \text{ for all } k.$$

In this case there may also be a decreasing bridge, so the functions φ_j are not necessarily building blocks. Since there is at worst one decreasing bridge of multiplicity 1, at most one φ_j is not a building block.

Once again, we work with a set $\mathcal{S} \subset \Sigma$ consisting of the functions φ_j for $j \notin \{h, h+1\}$, plus three more functions that are contained in the tropicalization of a pencil.

Proposition 9.18. *There is a pencil $W \subset V$ and functions φ_A, φ_B, and $\varphi_C \in \operatorname{trop}(W)$ with the following properties:*

(i) $s'_k(\varphi_A) = s'_k[h]$ *for all* $k < \ell$;
(ii) $s_k(\varphi_B) = s_k[h+1]$ *for all* $k > \ell$;
(iii) $s_k(\varphi_C) = s_k[h+1]$ *for all* $k \leq \ell$, *and* $s'_k(\varphi_C) = s_k[h]$ *for all* $k \geq \ell$;
(iv) $s_k(\varphi_\bullet) \in \{s_k[h], s_k[h+1]\}$ *and* $s'_k(\varphi_\bullet) \in \{s'_k[h], s'_k[h+1]\}$ *for all* k.

Proof. The argument is identical to the proof of Proposition 9.7. □

As in the previous case, the functions φ_A, φ_B, and φ_C can be written as tropical linear combinations of simpler functions in $R(D)$. We have the following analogue of Lemmas 9.8 and 9.10.

Lemma 9.19. *There are functions $\varphi_h^0, \varphi_{h+1}^0$, and $\varphi_h^\infty \in R(D)$ with the following properties:*

(i) $s_k(\varphi_h^0) = s_k[h]$ *and* $s'_k(\varphi_h^0) = s'_k[h]$ *for all* k;
(ii) $s_k(\varphi_{h+1}^0) = s_k[h+1]$ *and* $s'_k(\varphi_{h+1}^0) = s'_k[h+1]$ *for all* k;
(iii) $s_k(\varphi_h^\infty) = s_k[h]$, $s'_{k-1}(\varphi_h^\infty) = s'_{k-1}[h]$ *for all* $k \leq \ell$, *and* $s_k(\varphi_h^\infty) = s_k[h+1]$, $s'_{k-1}(\varphi_h^\infty) = s'_{k-1}[h+1]$ *for all* $k > \ell$.
(iv) *The function φ_A is a tropical linear combination of the functions φ_h^0 and φ_h^∞, where the two functions simultaneously achieve the minimum at a point to the right of γ_ℓ.*
(v) *The function φ_B is a tropical linear combination of the functions φ_{h+1}^0 and φ_h^∞, where the two functions simultaneously achieve the minimum at a point to the left of γ_ℓ.*
(vi) *The function φ_C is a tropical linear combination of the functions φ_h^0 and φ_{h+1}^0, where the two functions simultaneously achieve the minimum on the loop γ_ℓ where they agree.*

Proof. The construction of these three functions is identical to that of Lemma 9.8, and the verification of properties (iv)-(vi) is similar to the proof of Lemma 9.10. □

Proof of Theorem 9.1, case 3. We set

$$\mathcal{A}' = \{\varphi_i : i \neq h, h+1\} \cup \{\varphi_h^0, \varphi_{h+1}^0, \varphi_h^\infty\}.$$

If there are no decreasing bridges, then the set $\mathcal{A}'$ consists of building blocks, and the argument is identical to case 2.

On the other hand, if there is a decreasing bridge $\beta_{\ell'}$, then we combine the construction of case 2 with that from subcase 1b. Let $\mathcal{A}$ be the set of functions $\varphi \in R(D)$ with the following properties:

(i) there is a unique function $\varphi' \in \mathcal{A}'$ such that φ agrees with φ' on each connected component of $\Gamma \smallsetminus \beta_{\ell'}$, and
(ii) φ has constant slope on $\beta_{\ell'}$, equal to either $s'_{\ell'-1}[\tau'_{\ell'-1}(\varphi')]$ or $s_{\ell'}[\tau_{\ell'}(\varphi')]$.

The functions in $\mathcal{A}$ are building blocks, and by combining the arguments in the proofs of Lemmas 9.4 and 9.9, we see that $\mathcal{A}$ satisfies property (**A**). Note that each function in $\mathcal{A}'$ is a tropical linear combination of functions in $\mathcal{A}$, and hence so are φ_A, φ_B, and φ_C. Let $I \subset \beta_{\ell+1}$ be the interval where φ_A has slope $s'_\ell[h+1]$, and let $\widetilde{\mathcal{B}}'$ be the set of all pairwise sums of functions in $\mathcal{A}'$. Just as in case 2, we choose a subset $\mathcal{B}' \subset \widetilde{\mathcal{B}}'$, depending on the length of I, by omitting at most one function.

Let $\mathcal{B}$ be the set of pairwise sums of functions in $\mathcal{A}$ with the property that the associated pairwise sum of functions in $\mathcal{A}'$ is in $\mathcal{B}'$. Combining Lemmas 9.5 and 9.16, we see that $\mathcal{B}$ satisfies properties

(**B**) and (**B′**). We define $\mathcal{T}$ exactly as in case 2, and use Theorem 8.21 to construct the master template θ out of the functions in $\mathcal{B}$. We let θ' be the best approximation of θ by $\mathcal{B}'$, and then let ϑ be the best approximation of θ' by $\mathcal{T}$. Combining the arguments of case 2 and subcase 1b, we see that ϑ is a tropical independence. □

9.4. **Case 4: two switching loops.** We now consider the case where there are two switching loops, γ_ℓ and $\gamma_{\ell'}$, with $\ell < \ell'$. We write h and h' for the slopes that are switched by γ_ℓ and $\gamma_{\ell'}$, respectively. Note that both loops must have multiplicity 1. By our classification of switching loops in §6.8, we have

$$s'_\ell[i] = s_\ell[i] \text{ and } s'_{\ell'}[i] = s_{\ell'}[i] \text{ for all } i.$$

Moreover,

$$s_\ell[h+1] = s_\ell[h] + 1 \text{ and } s_{\ell'}[h'+1] = s_{\ell'}[h'] + 1.$$

Since $\rho = 2$ and we have two loops with positive multiplicity, by Proposition 6.18 there are no decreasing loops or bridges. This implies that there are only finitely many building blocks.

We break our analysis into several subcases, depending on the relationship between h and h'. By Lemma 6.22, for all $j \notin \{h, h+1, h', h'+1\}$, there is a function $\varphi_j \in \Sigma$ with

$$s_k(\varphi_j) = s_k[j] \text{ and } s'_k(\varphi_j) = s'_k[j] \text{ for all } k.$$

Because there are no decreasing bridges, the functions φ_j are building blocks.

9.4.1. *Subcase 4a: $h' \notin \{h-1, h, h+1\}$.* This is the simplest subcase because, roughly speaking, the two switching loops do not interact with one another. More precisely, there are functions φ_A, φ_B, and φ_C in Σ with slopes as defined in Proposition 9.18, and similarly, replacing ℓ with ℓ' and h with h', there are analogous functions φ'_A, φ'_B, and φ'_C in Σ. We may then define building blocks $\varphi^0_h, \varphi^0_{h+1}, \varphi^\infty_h, \varphi^0_{h'}, \varphi^0_{h'+1}$, and $\varphi^\infty_{h'}$ as in case 3, and set

$$\mathcal{A} = \{\varphi_i : i \neq h, h+1, h', h'+1\} \cup \{\varphi^0_h, \varphi^0_{h+1}, \varphi^\infty_h, \varphi^0_{h'}, \varphi^0_{h'+1}, \varphi^\infty_{h'}\}.$$

Our construction of the set $\mathcal{B}$ and the independence ϑ now follow the exact same steps as in case 3, treating each switching loop separately.

9.4.2. *Subcase 4b: $h' = h$.* We first identify a subset $\mathcal{S} \subset \Sigma$. It will consist of the functions φ_i for $i \notin \{h, h+1\}$, together with a subset of the functions illustrated in Figure 15.

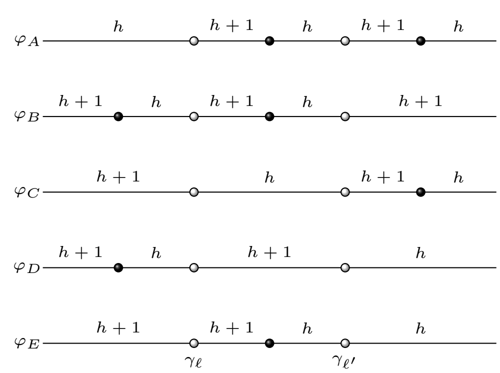

FIGURE 15. A schematic depiction of the five functions of Proposition 9.20.

Proposition 9.20. *There is a pencil $W \subset V$ and functions $\varphi_A, \varphi_B, \varphi_C, \varphi_D, \varphi_E \in \mathrm{trop}(W)$ with the following properties:*

(i) $s'_k(\varphi_A) = s'_k[h]$ *for all* $k < \ell$*;*
(ii) $s_k(\varphi_B) = s_k[h+1]$ *for all* $k > \ell'$*;*

(iii) $s_k(\varphi_C) = s_k[h+1]$ *for all* $k \leq \ell$ *and* $s'_k(\varphi_C) = s'_k[h]$ *for all* $\ell \leq k \leq \ell'$;
(iv) $s_k(\varphi_D) = s_k[h+1]$ *for all* $\ell < k \leq \ell'$ *and* $s'_k(\varphi_D) = s'_k[h]$ *for all all* $k \geq \ell'$;
(v) $s_k(\varphi_E) = s_k[h+1]$ *for all* $k \leq \ell$ *and* $s'_k(\varphi_E) = s'_k[h]$ *for all* $k \geq \ell'$;
(vi) $s_k(\varphi_\bullet) \in \{s_k[h], s_k[h+1]\}$ *and* $s'_k(\varphi_\bullet) \in \{s'_k[h], s'_k[h+1]\}$, *for all* k.

Proof. Let p and p' be points on X specializing to w_0 and $v_{g'+1}$, respectively. Consider a pencil W of functions in V that vanish to order at least h and $a_{r-(h+1)}(p)$ at p' and p, respectively. If $\varphi \in \mathrm{trop}(W)$, then $s_k(\varphi)$ is equal to either $s_k[h]$ or $s_k[h+1]$, exactly as in Proposition 9.7.

Choose φ_A and φ_B as in Proposition 9.7. To obtain φ_C, choose a function $\varphi \in \mathrm{trop}(W)$ such that $s_\ell(\varphi) = s_\ell[h+1]$, and choose a function $\varphi' \in \mathrm{trop}(W)$ such that $s'_\ell(\varphi') = s'_\ell[h]$. By adding a scalar to φ', we may assume that φ and φ' agree on γ_ℓ, and let $\varphi_C = \min\{\varphi, \varphi'\}$. The constructions of φ_D and φ_E are similar to that of φ_C. □

We now characterize two more functions in $R(D)$, depicted schematically in Figure 16.

Definition 9.21. There are functions $\psi, \psi' \in R(D)$, unique up to additive constants, with the following properties:

(i) $s_k(\psi) = s_k[h+1]$ for all $k \leq \ell'$, and $s'_k(\psi) = s'_k[h]$ for all $k \geq \ell'$;
(ii) $s_k(\psi') = s_k[h+1]$ for all $k \leq \ell$, and $s'_k(\psi') = s'_k[h]$ for all $k \geq \ell$;
(iii) $\mathrm{supp}(D + \mathrm{div}(\psi))$ contains $v_{\ell'}$ and $w_{\ell'}$;
(iv) $\mathrm{supp}(D + \mathrm{div}(\psi'))$ contains v_ℓ and w_ℓ.

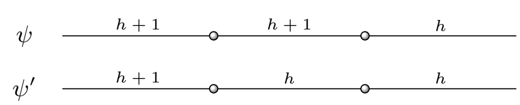

FIGURE 16. A schematic depiction of the functions ψ and ψ' from Definition 9.21.

Lemma 9.22. *Either* ψ *or* ψ' *is in* $\mathrm{trop}(W)$.

Proof. If $s_\ell(\varphi_D) = s_\ell[h+1]$, then $s_k(\varphi_D) = s_k[h+1]$ for all $k \leq \ell$, and we see that $\varphi_D = \psi$. Now, suppose that $s_\ell(\varphi_D) \neq s_\ell[h+1]$. Because W is a pencil, the functions φ_C, φ_D, and φ_E from Proposition 9.20 are tropically dependent. Since $s_\ell(\varphi_D) \neq s_\ell[h+1]$, in this dependence the functions φ_C and φ_E must achieve the minimum at v_ℓ. All three functions agree on the loop γ_ℓ, and since $s'_\ell(\varphi_C) = s'_\ell[h]$, it follows that one of the other two functions must also have slope $s'_\ell[h]$ along the bridge $\beta_{\ell+1}$. By definition, this function cannot be φ_D, so we must have $s'_\ell(\varphi_E) = s'_\ell[h]$. This implies that $s'_k(\varphi_E) = s'_k[h]$ for all $k \geq \ell$, hence $\varphi_E = \psi'$. □

Lemma 9.23. *If* ψ *is in* $\mathrm{trop}(W)$, *then* γ_ℓ *is not a switching loop for* $\mathrm{trop}(W)$. *Similarly, if* ψ' *is in* $\mathrm{trop}(W)$, *then* $\gamma_{\ell'}$ *is not a switching loop for* $\mathrm{trop}(W)$.

Proof. Suppose that $\psi \in \mathrm{trop}(W)$, and let $\varphi \in \mathrm{trop}(W)$ be a function with $s_\ell(\varphi) = s_\ell[h]$. Because W is a pencil, the functions φ, ψ, and φ_C are tropically dependent. Because $s_\ell(\varphi) = s_\ell[h]$, we see that in this dependence φ_C and ψ must achieve the minimum at w_ℓ. Since $s'_\ell(\varphi_C) = s'_\ell[h]$, it follows that one of the other two functions must also have slope $s'_\ell[h]$ along the bridge $\beta_{\ell+1}$. By definition, this function cannot be ψ, so it must be φ. The other case, where $\psi' \in \mathrm{trop}(W)$, is similar. □

Proof of Theorem 9.1, case 4b. If $\psi' \in \mathrm{trop}(W)$, we construct our independence ϑ as though $\gamma_{\ell'}$ is not a switching loop. The argument is the same as case 3, except with φ_C replaced by ψ'. Similarly, if $\psi \in \mathrm{trop}(W)$, we construct our independence ϑ as though γ_ℓ is not a switching loop. □

9.4.3. *Subcase 4c: $h' = h+1$.* We first identify a subset $\mathcal{S} \subseteq \Sigma$, consisting of the functions φ_i, for $i \notin \{h, h+1, h+2\}$, together with the functions $\varphi_A, \ldots, \varphi_E$ illustrated in Figure 17.

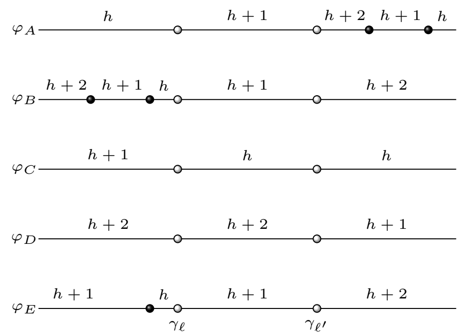

FIGURE 17. A schematic illustration of the five functions of Proposition 9.24.

Proposition 9.24. *There is a 3-dimensional subspace $W \subseteq V$, pencils $W_1, W_2 \subseteq W$, and functions $\varphi_A, \varphi_C \in \operatorname{trop}(W_1)$, $\varphi_B, \varphi_D \in \operatorname{trop}(W_2)$, and $\varphi_E \in \operatorname{trop}(W_1 \cap W_2)$ with the following properties:*

(i) $s'_k(\varphi_A) = s'_k[h]$ *for all* $k < \ell$;
(ii) $s_k(\varphi_B) = s_k[h+2]$ *for all* $k > \ell'$;
(iii) $s_k(\varphi_C) = s_k[h+1]$ *for all* $k \leq \ell$ *and* $s'_k(\varphi_C) = s'_k[h]$ *for all* $k \geq \ell$;
(iv) $s_k(\varphi_D) = s_k[h+2]$ *for all* $k \leq \ell'$, *and* $s'_k(\varphi_D) = s'_k[h+1]$ *for all all* $k \geq \ell'$;
(v) $s'_{k-1}(\varphi_E) = s_k(\varphi_E) = s_k[h+1]$ *for all* $\ell < k \leq \ell'$, *and* $s_k(\varphi_E) = s_k[h+2]$ *for all* $k > \ell'$;
(vi) $s_k(\varphi_\bullet) \in \{s_k[h], s_k[h+1], s_k[h+2]\}$ *and* $s'_k(\varphi_\bullet) \in \{s'_k[h], s'_k[h+1], s'_k[h+2]\}$ *for all* k.

Proof. As in the previous case, let p and p' be points on X specializing to w_0 and $v_{g'+1}$, respectively. Consider a 3-dimensional subspace $W \subseteq V$ of functions that vanish to order at least h and $a_{r-(h+2)}(p)$ at p' and p, respectively. Let $W_1 \subseteq W$ be a pencil of functions in V that vanish to order at least $a_{r-(h+1)}(p)$ at p. Similarly, let $W_2 \subseteq W$ be a pencil of functions in V that vanish to order at least $h+1$ at p'.

Choose φ_A and φ_B as in Proposition 9.7. The functions φ_C and φ_D are constructed in a manner similar to that of Proposition 9.20. Finally, let φ_E be a function in $\operatorname{trop}(W_1 \cap W_2)$.

By arguments analogous to the proofs of Propositions 9.7 and 9.20, the functions $\varphi_A, \varphi_B, \varphi_C$, and φ_D have the required slopes. We now describe the slopes of φ_E. Since $\varphi_E \in \operatorname{trop}(W_1)$, we see that $s_k(\varphi_E) \in \{s_k[h], s_k[h+1]\}$ for all $\ell < k \leq \ell'$, and since $\varphi_E \in \operatorname{trop}(W_2)$, we see that $s_k(\varphi_E) \in \{s_k[h+1], s_k[h+2]\}$ for all $\ell < k \leq \ell'$. It follows that $s_k(\varphi_E) = s_k[h+1]$ for all $\ell < k \leq \ell'$. The same argument shows that $s'_k(\varphi_E) = s'_k[h+1]$ for all $\ell \leq k < \ell'$. Moreover, the three functions φ_B, φ_D, and φ_E in $\operatorname{trop}(W_2)$ are tropically dependent, and the dependence is illustrated schematically in Figure 18. A priori, one might expect there to be a region to the right of $\gamma_{\ell'}$ where φ_D and φ_E agree in this dependence, but our assumptions on edge lengths preclude this. Specifically, since φ_D has higher slope than φ_B and φ_E along the bridge $\beta_{\ell+1}$, it cannot obtain the minimum to the right of this bridge. It follows that $s_k(\varphi_E) = s_k[h+2]$ for all $k > \ell'$. □

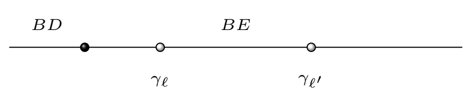

FIGURE 18. The function φ_B is a tropical linear combination of φ_D and φ_E.

Lemma 9.25. *The functions φ_C and φ_D do not agree on any loop. Moreover, for any pair $k' \leq k$, with $k \neq \ell$ and $k' \neq \ell'$, either φ_A or φ_E does not agree with φ_C on γ_k, nor with φ_D on $\gamma_{k'}$.*

Proof. Note that the white dots in Figure 17 representing γ_ℓ and $\gamma_{\ell'}$ divide the graph into 3 regions. Identify the regions containing $\gamma_{k'}$ and γ_k. For each of the 6 possibilities, one of the functions φ_A or φ_E disagrees with φ_D on the region containing $\gamma_{k'}$ and with φ_C on the region containing γ_k. For example, if $k' \leq k \leq \ell$, then $s'_{k'}(\varphi_A) \neq s'_{k'}(\varphi_D)$, so φ_A does not agree with φ_D on $\gamma_{k'}$, and $s'_k(\varphi_A) \neq s'_k(\varphi_C)$, so φ_A does not agree with φ_C on γ_k. The other 5 cases are similar. □

Lemma 9.26. *The set $\mathcal{A}$ of all building blocks satisfies property* (**A**).

Proof. The proof is identical to case 3 in the special case where there are no decreasing bridges. □

Note that the three functions φ_A, φ_C, and φ_E in $\mathrm{trop}(W_1)$ are tropically dependent; the dependence is illustrated schematically in Figure 19. Let y and y' be points, to the right and left of γ_ℓ, respectively, where all three functions simultaneously achieve the minimum in this dependence.

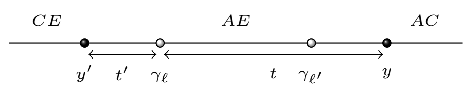

FIGURE 19. The dependence satisfied by φ_A, φ_C, and φ_E in case 4c.

Definition 9.27. Let t be the distance, measured along the bridges and bottom edges, from w_ℓ to y, and similarly let t' be the distance from v_ℓ to y'.

Just as in case 2, t' is an increasing piecewise affine function of t. We now describe how to choose the set $\mathcal{B}$, depending on the parameter t, in a manner similar to Definition 9.15.

Definition 9.28. Let $\widetilde{\mathcal{B}}$ be the set of pairwise sums of elements of $\mathcal{A}$. Suppose that there are two indices j and j' such that $s_\ell(\theta) = s'_\ell[h] + s'_\ell[j] = s'_\ell[h] + s'_\ell[j'] + 1$. Then

(i) if $t_1 < m_\ell$, let $\widehat{\mathcal{B}} = \widetilde{\mathcal{B}} \smallsetminus \{\varphi + \varphi_{j'} : s_{\ell+1}(\varphi) = s_\ell(\varphi) + 1 = s_{\ell+1}[h+1]\}$;
(ii) if $t_1 \geq m_\ell$, let $\widehat{\mathcal{B}} = \widetilde{\mathcal{B}} \smallsetminus \{\varphi + \varphi_j : s_{\ell+1}(\varphi) = s_\ell(\varphi) = s_{\ell+1}[h]\}$.

Otherwise, let $\widehat{\mathcal{B}} = \widetilde{\mathcal{B}}$.

Now, if there are indices i, i' such that $s_{\ell'}(\theta) = s'_{\ell'}[h+1] + s'_{\ell'}[i] = s'_{\ell'}[h+1] + s'_{\ell'}[i'] + 1$, let $\mathcal{B} = \widehat{\mathcal{B}} \smallsetminus \{\varphi + \varphi_i : s_{\ell'+1}(\varphi) = s_{\ell'}(\varphi) = s_{\ell'+1}[h+1]\}$. Otherwise, let $\mathcal{B} = \widehat{\mathcal{B}}$.

Note that the point where φ_B, φ_D, and φ_E simultaneously achieve the minimum in Figure 18 is to the left of γ_ℓ. The distance from $v_{\ell'}$ to this point is therefore larger than $m_{\ell'}$, and the construction of $\mathcal{B}$ from $\widehat{\mathcal{B}}$ is analogous to the construction of $\widehat{\mathcal{B}}$ from $\widetilde{\mathcal{B}}$ in case (ii).

The set $\mathcal{B}$ satisfies properties (**B**) and (**B**$'$) just as in Lemma 9.16. We may therefore construct the master template θ. Our choice of $\mathcal{T}$ is very similar to case 2. Specifically, if $i, j \notin \{h, h+1, h+2\}$, then we put $\varphi_{ij}, \varphi_C + \varphi_j$, and $\varphi_D + \varphi_j$ in $\mathcal{T}$. We then put one of $\varphi_A + \varphi_j$ or $\varphi_E + \varphi_j$ in $\mathcal{T}$, depending on where the best approximation of θ by $\varphi_C + \varphi_j$ and $\varphi_D + \varphi_j$ achieves equality. We use Lemma 9.25 to choose this function, as follows. The function $\varphi_C + \varphi_j$ achieves equality on the region where some pairwise sum of building blocks $\psi \in \mathcal{B}$ achieves the minimum, and $\varphi_D + \varphi_j$ achieves equality on the region where some pairwise sum of building blocks $\psi' \in \mathcal{B}$ achieves the minimum. By Lemma 9.25, ψ and ψ' are not assigned to the same loop or bridge, and one of the functions $\varphi_A + \varphi_j$ or $\varphi_E + \varphi_j$ disagrees with both $\varphi_C + \varphi_j$ on the loop or bridge where ψ is assigned and with $\varphi_D + \varphi_j$ on the loop or bridge where ψ' is assigned. We put this function in $\mathcal{T}$.

Similarly, we include six pairwise sums of elements of $\{\varphi_A, \varphi_C, \varphi_D, \varphi_E\}$ in $\mathcal{T}$. In all cases, we put $\varphi_C + \varphi_C, \varphi_C + \varphi_D$, and $\varphi_D + \varphi_D$ in $\mathcal{T}$. Then, depending on where the best approximation of θ by these functions achieves equality, we put one of $\varphi_A + \varphi_C$ or $\varphi_C + \varphi_E$ in $\mathcal{T}$, and one of $\varphi_A + \varphi_D$ or $\varphi_D + \varphi_E$ in $\mathcal{T}$. Finally, depending on where the best approximation of θ by these two functions achieves equality, we put one of $\varphi_A + \varphi_A, \varphi_A + \varphi_E$, or $\varphi_E + \varphi_E$ in $\mathcal{T}$.

Proof of Theorem 9.1, case 4c. The proof of this subcase is very similar to that of case 2. The construction of $\mathcal{B}$ guarantees that, in the best approximation, each function in $\mathcal{T}$ achieves equality on a region where some function in $\mathcal{B}$ achieves the minimum in θ. Lemma 9.25 then shows that no two of these functions achieve the minimum on the same loop or bridge. □

9.4.4. *Subcase 4d: $h' = h - 1$.* In the previous three cases, our analysis reduced to the study of the tropicalizations of certain pencils. In this last case, we instead reduce to a rank 2 linear series. Nevertheless, the arguments are of a similar flavor, with just a few more combinatorial possibilities. As in the previous cases, we begin by describing the subset $\mathcal{S} \subset \Sigma$. It consists of the functions φ_i, for $i \notin \{h-1, h, h+1\}$, together the functions $\varphi_A, \dots, \varphi_H$ illustrated in Figures 20 and 22.

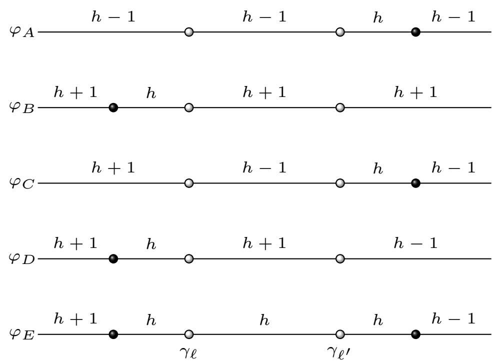

FIGURE 20. A schematic depiction of the five functions of Proposition 9.29.

Proposition 9.29. *There is a 3-dimensional subspace $W \subseteq V$ and functions $\varphi_A, \varphi_B, \varphi_C, \varphi_D, \varphi_E \in$ trop(W) with the following properties:*

(i) $s'_k(\varphi_A) = s'_k[h-1]$ *for all* $k < \ell'$;
(ii) $s_k(\varphi_B) = s_k[h+1]$ *for all* $k > \ell$;
(iii) $s_k(\varphi_C) = s_k[h+1]$ *for all* $k \leq \ell$ *and* $s'_k(\varphi_C) = s'_k[h-1]$ *for all* $\ell \leq k < \ell'$;
(iv) $s_k(\varphi_D) = s_k[h+1]$ *for all* $\ell < k \leq \ell'$, *and* $s'_k(\varphi_D) = s'_k[h-1]$ *for all all* $k \geq \ell'$;
(v) $s'_{k-1}(\varphi_E) = s_k(\varphi_E) = s_k[h]$ *for all* $\ell < k \leq \ell'$;
(vi) $s_k(\varphi_\bullet) \in \{s_k[h-1], s_k[h], s_k[h+1]\}$ *and* $s'_k(\varphi_\bullet) \in \{s'_k[h-1], s'_k[h], s'_k[h+1]\}$, *for all* k.

Proof. As before, let p and p' be points on X specializing to w_0 and $v_{g'+1}$, respectively. Let $W \subseteq V$ be a 3-dimensional subspace of functions that vanish to order at least $h-1$ at p' and order at least $a_{r-(h+1)}(p)$ at p. Choose φ_A and φ_B as in Proposition 9.7. The functions φ_C and φ_D are constructed as in Proposition 9.20. Finally, let φ_E be a function in trop(W) with $s_{\ell+1}(\varphi_E) \leq s_{\ell+1}[h]$ and $s_{\ell'}(\varphi_E) \geq s_{\ell'}[h]$. □

Lemma 9.30. *Either* $s'_k(\varphi_A) = s'_k[h-1]$ *for all* $k \geq \ell'$, *or* $s'_k(\varphi_E) = s'_k[h-1]$ *for all* $k \geq \ell'$.

Proof. Because W is 3-dimensional, the functions $\varphi_A, \varphi_B, \varphi_D$, and φ_E are tropically dependent. Only φ_B and φ_D have the same slope along $\beta_{\ell'}$, thus these two achieve the minimum at $v_{\ell'}$. Because of this, φ_D must also achieve the minimum at $w_{\ell'}$. Since it has slope $s_{\ell'}[h-1]$ along $\beta_{\ell'+1}$, there must be a second function among these four with this same slope along $\beta_{\ell'+1}$. This function can only be φ_A or φ_E. □

Lemma 9.31. *Either* $s_k(\varphi_B) = s_k[h+1]$ *for all* $k \leq \ell$, *or* $s_k(\varphi_E) = s_k[h+1]$ *for all* $k \leq \ell$.

Proof. This is similar to the proof of Lemma 9.30, using the functions $\varphi_A, \varphi_B, \varphi_C$, and φ_E. □

Lemmas 9.30 and 9.31 together produce 4 possible cases. In all but one of these cases, trop(W) switches only one loop.

Lemma 9.32. *If $s'_k(\varphi_A) = s'_k[h-1]$ for all $k \geq \ell'$, then $\gamma_{\ell'}$ is not a switching loop for* trop(W). *Similarly, if $s_k(\varphi_B) = s_k[h+1]$ for all $k \leq \ell$, then γ_ℓ is not a switching loop for* trop(W).

Proof. We consider the case where $s'_k(\varphi_A) = s'_k[h-1]$ for all $k \geq \ell'$. The other case is similar. Let $\varphi \in$ trop(W) be a function with $s_{\ell'+1}(\varphi) = s_{\ell'+1}[h]$. Because W is 3-dimensional, the functions $\varphi, \varphi_A, \varphi_B$ and φ_E are tropically dependent. Because only φ and φ_E have the same slope on $\beta_{\ell'+1}$, in this dependence they must achieve the minimum at $w_{\ell'}$. Because of this, φ_E achieves the minimum at $v_{\ell'}$ as well, hence the minimum has slope at least $s_{\ell'}[h]$ along $\beta_{\ell'}$. Because this slope must be obtained twice, and the three functions φ_A, φ_B, and φ_E have distinct slopes there, we see that $s_{\ell'}(\varphi) \geq s_{\ell'}[h]$. □

If trop(W) switches only one loop, then the argument is identical to case 3. For the remainder of this section, we assume that there exists $k' \geq \ell'$ such that $s'_{k'}(\varphi_A) = s'_{k'}[h]$, and there exists $k \leq \ell$ such that $s_k(\varphi_B) = s_k[h+1]$. By Lemmas 9.30 and 9.31, this implies that the slopes of φ_E are as pictured in Figure 21.

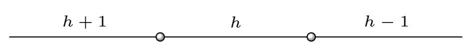

FIGURE 21. A schematic depiction of the function φ_E when trop(W) switches both loops.

We now describe additional functions in trop(W). These functions are illustrated in Figure 22.

Proposition 9.33. *There exist functions $\varphi_F, \varphi_G, \varphi_H \in$* trop$(W)$ *with the following properties:*

(i) $s'_k(\varphi_F) = s'_k[h]$ *for all* $k \leq \ell$, *and* $s_\ell(\varphi_F) = s_\ell[h+1]$;
(ii) $s'_{\ell'}(\varphi_G) = s'_{\ell'}[h-1]$ *and* $s_k(\varphi_G) = s_k[h]$ *for all* $k > \ell'$;
(iii) *either*
  (a) $s'_k(\varphi_H) = s'_k[h]$ *for all* $k \leq \ell$ *and* $s_k(\varphi_H) = s_k[h]$ *for all* $k > \ell'$, *or*
  (b) $s'_k(\varphi_H) = s'_k[h]$ *for all* $k \leq \ell$ *and* $s_k(\varphi_H) = s_k[h+1]$ *for all* $\ell < k \leq \ell'$, *or*
  (c) $s'_k(\varphi_H) = s'_k[h-1]$ *for all* $\ell \leq k < \ell'$, *and* $s_k(\varphi_H) = s_k[h]$ *for all* $k > \ell'$.

.

Proof. Let $f_A, f_B, f_E \in W$ be functions tropicalizing to φ_A, φ_B, and φ_E, respectively. We let φ_F be the tropicalization of a function in the pencil spanned by f_B and f_E with the property that $s'_0(\varphi_F) \neq s'_0[h+1]$. Similarly, we let φ_G be the tropicalization of a function in the pencil spanned by f_A and f_E with the property that $s_{g'+1}(\varphi_G) \neq s_{g'+1}[h-1]$. We let φ_H be a function in trop(W) such that $s'_0(\varphi_H) \leq s'_0[h]$ and $s_{g'+1}(\varphi_H) \geq s_{g'+1}[h]$.

To see that the functions have the required slopes, we make use of various dependences between them and the functions $\varphi_A, \varphi_B, \varphi_E$. Specifically, because the functions φ_B, φ_E, and φ_F are tropicalizations of functions in a pencil, they are tropically dependent. The dependence between them is very similar to the dependence between φ_A, φ_B, and φ_C in case 2, and is depicted in the top line of Figure 23. In this dependence, φ_B and φ_F agree in a neighborhood of γ_ℓ, which determines the slopes of φ_F on the bridges to either side of this loop.

Similarly, the functions φ_A, φ_E, and φ_G are tropically dependent, and the dependence between them is illustrated in the bottom line of Figure 23. The functions $\varphi_A, \varphi_B, \varphi_E$, and φ_H also satisfy a dependence. There are three possibilities for this dependence, as shown in Figure 24. □

Lemma 9.34. *The functions φ_A and φ_B do not agree on any loop. Moreover, for any pair $k' \leq k$ with $k' \neq \ell'$ and $k \neq \ell$, one of the four functions $\varphi_E, \varphi_F, \varphi_G, \varphi_H$ does not agree with φ_B on $\gamma_{k'}$, nor with φ_A on γ_k.*

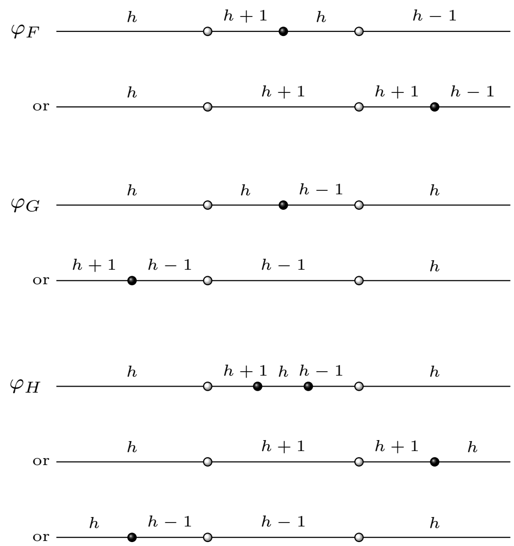

FIGURE 22. A schematic depiction of the three functions of Proposition 9.33.

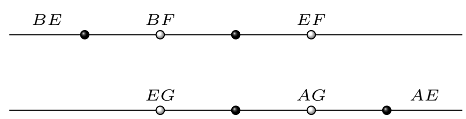

FIGURE 23. Dependences between the functions $\varphi_A, \varphi_B, \varphi_E, \varphi_F$, and φ_G.

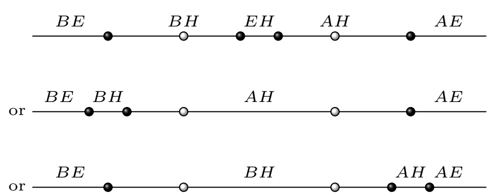

FIGURE 24. Possibilities for the dependence between $\varphi_A, \varphi_B, \varphi_E$, and φ_H.

Proof. The proof is similar to that of Lemma 9.25. □

We choose the set $\mathcal{A}$ satsifying property (**A**) and the set $\mathcal{B}$ satisfying properties (**B**) and (**B′**) exactly as in 4c, and can then construct the master template θ. The choice of $\mathcal{T}$ is also similar to case 4c. First, if $i, j \notin \{h-1, h, h+1\}$, then we put $\varphi_{ij}, \varphi_A + \varphi_j$, and $\varphi_B + \varphi_j$ in $\mathcal{T}$. Then $\varphi_A + \varphi_j$ achieves equality on the region where some pairwise sum of building blocks $\psi \in \mathcal{B}$ achieves the minimum, and $\varphi_B + \varphi_j$ achieves equality on the region where some pairwise sum of building blocks ψ' achieves the minimum. By Lemma 9.34, ψ and ψ' are not assigned to the same loop or bridge, and one of the four functions $\varphi_E + \varphi_j, \varphi_F + \varphi_j, \varphi_G + \varphi_j$, or $\varphi_H + \varphi_j$ disagrees with both $\varphi_A + \varphi_j$ on the loop or bridge where ψ is assigned, and with $\varphi_B + \varphi_j$ on the loop or bridge where ψ' is assigned. We put this function in $\mathcal{T}$.

Proof of Theorem 9.1, case 4d. The proof is identical to that of case 4c, using Lemma 9.34 in place of Lemma 9.25. □

10. Effectivity of the virtual classes

We retain the notation from previous sections, fixing $r = 6$, $g = 21 + \rho$, $d = 24 + \rho$, and $\rho \in \{1, 2\}$. In §3 we defined an open substack $\widetilde{\mathfrak{M}}_g$ of the moduli stack of stable curves, a stack $\widetilde{\mathfrak{G}}^r_d$ of generalized limit linear series of rank r and degree d over $\widetilde{\mathfrak{M}}_g$, and a morphism of vector bundles $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ over $\widetilde{\mathfrak{G}}^r_d$, whose degeneracy locus is denoted by $\mathfrak{U}$.

In §9 we used the method of tropical independence to prove Theorem 1.3, establishing the Strong Maximal Rank Conjecture for g, r, and d. As a consequence, we know that the push forward $\sigma_*[\mathfrak{U}]^{\mathrm{virt}}$ under the proper forgetful map $\sigma : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathcal{M}}_g$ is a divisor, not just a divisor class. We now proceed to prove Theorem 1.4, which says that $\mathfrak{U}$ is generically finite over each component of this divisor. This implies that $\sigma_*[\mathfrak{U}]^{\mathrm{virt}}$ is effective. By Theorem 1.2, the slope of this effective divisor is less than $\frac{13}{2}$, and it follows that $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type.

10.1. **Multiplication maps with ramification.** To study the fibers of $\sigma_{|\mathfrak{U}}$ over singular curves, we consider linear series $\ell = (L, V)$ of degree d and rank r on a pointed curve (X, p) of genus $g' \leq g$ that satisfy a ramification condition at p. More precisely, as in §§8-9, we consider the cases where

(i) $g' = g$,
(ii) $g' = g - 1$ and $a^\ell_1(p) \geq 2$, or
(iii) $g' = g - 2$ and either $a^\ell_1(p) \geq 3$ or $a^\ell_0(p) + a^\ell_2(p) \geq 5$.

We deduced Theorem 1.3 from the case of Theorem 9.1 where $g' = g$. The cases where g' is equal to $g-1$ or $g-2$ have the following analogous consequences involving multiplication maps for linear series with ramification on a general pointed curve of genus g'.

Theorem 10.1. *Let X be a general curve of genus $g' = 20 + \rho$ and let $p \in X$ be a general point. Then the multiplication map*

$$\phi_\ell : \mathrm{Sym}^2 V \to H^0(X, L^{\otimes 2})$$

is injective for all linear series $\ell = (L, V) \in G^6_{24+\rho}(X)$ satisfying the vanishing condition $a^\ell_1(p) \geq 2$.

Theorem 10.2. *Let X be a general curve of genus $g' = 19 + \rho$ and let $p \in X$ be a general point. Then the multiplication map*

$$\phi_\ell : \mathrm{Sym}^2 V \to H^0(X, L^{\otimes 2})$$

is injective for all linear series $\ell = (L, V) \in G^6_{24+\rho}(X)$ satisfying either of the vanishing conditions:

$$a^\ell_1(p) \geq 3 \quad or \quad a^\ell_0(p) + a^\ell_2(p) \geq 5.$$

10.2. **Effectivity via numerical vanishing.** For the remainder of the section, suppose $Z \subseteq \overline{\mathcal{M}}_g$ is an irreducible divisor and that $\sigma_{|\mathfrak{U}}$ has positive dimensional fibers over the generic point of Z. Our strategy for proving Theorem 1.4 is to show, using the vanishing criterion from §2.3, that $[Z] = 0$ in $CH^1(\overline{\mathcal{M}}_g)$. This is impossible, since $\overline{\mathcal{M}}_g$ is projective, and hence no such Z exists. To apply the vanishing criterion, we must show:

(1) D is the closure of a divisor in $\mathcal{M}_g$,
(2) $\jmath_2^*(D) = 0$,
(3) D does not contain any codimension 2 stratum $\Delta_{2,j}$, and
(4) if g is even then $\jmath_3^*(D)$ is a nonnegative combination of the classes $[\overline{\mathcal{W}}_3]$ and $[\overline{\mathcal{H}}_3]$ on $\overline{\mathcal{M}}_{3,1}$.

The only irreducible divisors on $\widetilde{\mathcal{M}}_g$ in the complement of $\mathcal{M}_g$ are Δ_0° and Δ_1°. Therefore, the fact that Z must be the closure of a divisor in $\mathcal{M}_g$ is a consequence of the following proposition.

Proposition 10.3. *The image of the degeneracy locus $\mathfrak{U}$ does not contain Δ_0° or Δ_1°.*

Proof. Let $[X, p] \in \mathcal{M}_{g-1,1}$ be a general pointed curve and consider the curve Y obtained by gluing a nodal rational curve E_∞ to X at the point p. Note that $[Y] \in \Delta_0^\circ \cap \Delta_1^\circ$. The X-aspect of a generalized limit linear series of type $\mathfrak{g}^r_d$ on Y is a linear series $\ell \in G^r_d(X)$ satisfying the condition $a^\ell_1(p) \geq 2$. Then Theorem 10.1 implies that $[Y] \notin \sigma(\mathfrak{U})$. □

This verifies that Z satisfies (1). In the proofs that Z satisfies (2)-(4), we use the following lemma.

Lemma 10.4. *If $[X] \in Z$ and $p \in X$ then there is a linear series $\ell \in G^r_d(X)$ that is ramified at p such that ϕ_ℓ is not injective.*

Proof. If $[X] \in Z$, then there are infinitely many linear series $\ell \in G^r_d(X)$ for which ϕ_ℓ fails to be injective. By [Sch91, Lemma 2.a], at least one such linear series is ramified at the point p. □

10.3. **Pulling back to $\overline{\mathcal{M}}_{2,1}$.** In order to verify (2), we now consider the preimage of Z under the map $\jmath_2 \colon \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_g$ obtained by attaching an arbitrary pointed curve of genus 2 to a fixed general pointed curve (X, p) of genus $g-2$.

Lemma 10.5. *The preimage $\jmath_2^{-1}(Z)$ is contained in the Weierstrass divisor $\overline{\mathcal{W}}_2$ in $\overline{\mathcal{M}}_{2,1}$.*

Proof. Let C be an arbitrary curve of genus 2 and (abusing notation slightly) let $p \in C$ be a non-Weierstrass point. If $[Y] := [X \cup_p C]$ is in Z, then it is in the closure of the generic point $[Y_t]$ of a one-parameter family in $\sigma(\mathfrak{U})$. Since $[Y_t]$ is in $\sigma(\mathfrak{U})$, there is a linear series ℓ_t on Y_t for which the multiplication map ϕ_{ℓ_t} is not injective. Hence there is a limit linear series ℓ on Y such that the multiplication map on each aspect of ℓ is not injective.

We claim that the X-aspect ℓ_X of any limit linear series ℓ on $X \cup_p C$ satisfies one of the ramification conditions $a_1^{\ell_X}(p) \geq 3$ or $a_0^{\ell_X}(p) + a_2^{\ell_X}(p) \geq 5$. Suppose both inequalities fail. By failure of the first inequality and the definition of a limit linear series, we have $a_{r-1}^{\ell_C}(p) \geq d-2$. Since $p \in C$ is not a Weierstrass point, this forces $a_r^{\ell_C}(p) = d-1$. By the definition of a limit linear series, this gives $a_0^{\ell_X}(p) \geq 1$ and hence $a_2^{\ell_X}(p) \geq 3$. By failure of the second inequality, we have $a_2^{\ell_X}(p) = 3$, and hence $a_{r-2}^{\ell_C}(p) \geq d-3$. Then $\dim \left|\ell_C(-(d-3)p)\right| \geq 2$, which contradicts Riemann-Roch. This proves the claim, and the result then follows from Theorem 10.2. □

In the proof of the next proposition, and for the remainder of the paper, our arguments use tropical and nonarchimedean analytic geometry. All of the curves and maps that appear are defined over our fixed nonarchimedean field K.

Proposition 10.6. *We have $\jmath_2^*(Z) = 0$.*

Proof. Since the Weierstrass divisor $\overline{\mathcal{W}}_2$ is irreducible, we only need to show that $\jmath_2^{-1}(Z)$ does not contain $\overline{\mathcal{W}}_2$. To do this, we exhibit a point in the Weierstrass divisor that does not lie in $\jmath_2^{-1}(Z)$, as follows. Let Γ be a chain of $g-2$ loops with bridges whose edge lengths are admissible in the sense of Definition 6.10, and let Y be a smooth curve of genus $g-2$ over K whose skeleton is Γ. Let $p \in Y$ be a point specializing to the left endpoint of Γ. We consider the map $\jmath_2 \colon \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_g$ obtained by attaching the pointed curve (Y, p) to an arbitrary stable pointed curve of genus 2.

Let Y' be a smooth curve of genus 2 over K whose skeleton Γ' is a chain of 2 loops connected by a bridge. The tropicalization of the Weierstrass points on Y' are known, and do not depend on the choice of curve with this skeleton. See, e.g., [Ami14] or [JL18, Theorem 1.1]. In particular, there is a Weierstrass point $p \in Y'(K)$ whose specialization is a 2-valent point on the right loop. Let $Y'' := Y \cup_p Y'$. The skeleton of Y'' is obtained from Γ and Γ' by attaching infinitely long bridges at the specializations of p, and then gluing the infinitely far endpoints to each other, as in Figure 25.[3] Note that $[Y''] \in \jmath_2(\overline{\mathcal{W}}_2)$. We will show that $[Y''] \notin Z$.

[3]We recall that the topological space Y^{an} is obtained from its skeleton Γ by attaching an $\mathbb{R}$-tree rooted at each point. The K-points of Y naturally correspond to the leaves of these $\mathbb{R}$-trees, and each leaf is infinitely far from the skeleton Γ, in the natural metric on $Y^{\mathrm{an}} \smallsetminus Y(K)$. Hence, the analytification of the nodal curve $Y \cup Y'$ contains a skeleton which is the union of Γ, Γ', and the infinite length paths from Γ and Γ', respectively, to the node p. See, for instance, [ACP15, §8.3].

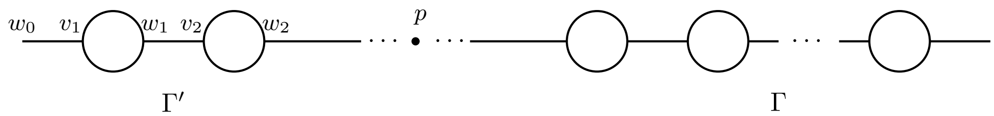

FIGURE 25. The skeleton of Y'' is the union of the skeletons Γ' and Γ of Y' and Y, respectively, and the unique embedded paths from these skeletons to p.

If $[Y''] \in Z$, then Z contains smooth curves X whose skeletons are arbitrarily close to the skeleton of Y'' in the natural topology on $\overline{M}_g^{\mathrm{trop}}$. In particular, for each integer $N > 0$, there is an $[X] \in Z$ with skeleton a chain of loops $\widetilde{\Gamma}_X$ with edge lengths as follows. We label the vertices and edges of $\widetilde{\Gamma}_X$ just as we have done for other chains of loops with bridges. Then the bridge β_3 has length greater than N, and each other edge has length within $\frac{1}{N}$ of the corresponding edge in Γ and Γ'. The metric graph $\widetilde{\Gamma}_X$ is similar to the skeleton pictured in Figure 25, except that the doubly infinite bridge containing p is replaced by an ordinary finite bridge that is much longer than all other edges. We divide $\widetilde{\Gamma}_X$ into two subgraphs $\widetilde{\Gamma}'$ and $\widetilde{\Gamma}$, to the left and right, respectively, of the midpoint of the long bridge β_3. (These subgraphs are very similar to Γ' and Γ, respectively.) Let $q \in X$ be a point specializing to v_{g+1}. Since $[X] \in Z$, by Lemma 10.4 there is a linear series in the degeneracy locus over X that is ramified at q. We now show that this impossible.

Let $\ell = \big(L, V\big) \in G^6_{g+3}(X)$ be a linear series ramified at q. We may assume that $L = \mathcal{O}(D_X)$, where $D = \mathrm{Trop}(D_X)$ is a break divisor, and consider $\Sigma = \mathrm{trop}(V)$. We will show that there are 28 tropically independent pairwise sums of functions in Σ using a variant of the arguments in §§8-9. It follows that the multiplication map ϕ_ℓ is injective, and hence $[X]$ cannot be in Z.

To produce 28 tropically independent pairwise sums of functions in Σ, following the methods of §§8-9, we first consider the slope sequence along the long bridge β_3. First, suppose that either $s_3[5] \leq 2$ or $s_3[4] + s_3[6] \leq 5$. In this case, even though the restriction of Σ to $\widetilde{\Gamma}$ is not the tropicalization of a linear series on a pointed curve of genus $g-2$ with prescribed ramification, it satisfies all of the combinatorial properties of the tropicalization of such a linear series. The proof of Theorem 9.1 then goes through verbatim, yielding a tropical linear combination of 28 functions in Σ such that each function achieves the minimum uniquely at some point of $\widetilde{\Gamma} \subset \widetilde{\Gamma}_X$.

For the remainder of the proof, we therefore assume that $s_3[5] \geq 3$ and $s_3[4] + s_3[6] \geq 6$. Since $\deg D_{|\widetilde{\Gamma}'} = 5$, we see that $s_3[6] \leq 5$. Moreover, since the divisor $D_{|\widetilde{\Gamma}'} - s_3[5]w_2$ has positive rank on $\widetilde{\Gamma}'$, and no divisor of degree 1 on $\widetilde{\Gamma}'$ has positive rank, $s_3[5]$ must be exactly 3. Since the canonical class is the only divisor class of degree 2 and rank 1 on $\widetilde{\Gamma}'$, we see that $D_{|\widetilde{\Gamma}'} \sim K_{\widetilde{\Gamma}'} + 3w_2$. This yields an upper bound on each of the slopes $s_3[i]$, and these bounds determine the slopes for $i \geq 3$:

$$s_3[6] = 5,\ s_3[5] = 3,\ s_3[4] = 1,\ s_3[3] = 0.$$

Moreover, we must have $s'_2[i] = s_3[i]$ for $3 \leq i \leq 6$. Since ℓ is ramified at q, we also have $s_{g+1}[6] \geq 7$. By Proposition 6.18, these conditions together imply that the sum of the multiplicities of all loops and bridges on $\widetilde{\Gamma}$ is at most 2.

To construct an independence on $\widetilde{\Gamma}_X$, we first construct an independence among 5 functions on $\widetilde{\Gamma}'$. The construction is analogous to that in §9.3, with the second loop of $\widetilde{\Gamma}'$ playing the role of a switching loop. The details are as follows.

For $i = 5, 6$, there is a function $\varphi_i \in \Sigma$ such that

$$s_k(\varphi_i) = s_k[i] \text{ for all } k \leq 3 \text{ and } s'_k(\varphi_i) = s'_k[i] \text{ for all } k \leq 2.$$

We also have φ_B, φ_C in Σ (analogous to the similarly labeled functions in §9.3) satisfying:

$$s'_0(\varphi_C) = s_1(\varphi_C) = s'_1(\varphi_C) = s_2(\varphi_C) = s_1[4] = 1, s'_2(\varphi_C) = s_3(\varphi_C) = s_3[3] = 0,$$

$$s'_2(\varphi_B) = s_3(\varphi_B) = s_3[4] = 1.$$

Moreover, the slope of φ_B at any point along the first 3 bridges is either 0 or 1. Note in particular that all of the functions ψ in the set $\{\varphi_{66}, \varphi_{56}, \varphi_{55}, \varphi_B + \varphi_6\}$ satisfy $s_3(\psi) \geq 6$, and $s_3(\varphi_C + \varphi_6) = 5$.

On the first bridge and first loop, we build an independence among the functions $\varphi_{66}, \varphi_{56}, \varphi_{55}$, and $\varphi_B + \varphi_6$ as in Figure 26. Since all 4 of these functions have slope at least 6 along the very long bridge β_3, and $\varphi_C + \varphi_6$ has slope 5, we may set the coefficient of $\varphi_C + \varphi_6$ so that it obtains the minimum at some point of the very long bridge, but not at any point of the first two loops or bridges.

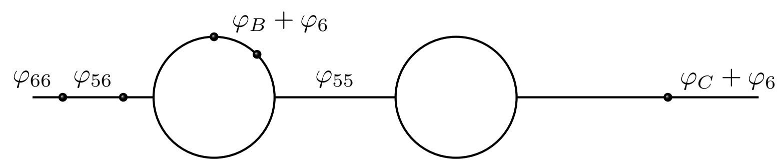

FIGURE 26. An independence on $\widetilde{\Gamma}'$

We now construct an independence among 23 pairwise sums of functions in Σ restricted to $\widetilde{\Gamma}$. By Theorem 6.18, our computation of the slopes $s'_2(\Sigma)$, together with the fact that ℓ is ramified at q, imply that the sum of the multiplicities of all loops and bridges on $\widetilde{\Gamma}$ is at most 2. Just as in §8.4, but restricting to $\widetilde{\Gamma}$, we associate a sequence of partitions to Σ, use these partitions to characterize the integers z and z', and thereby define the slopes $s_k(\theta)$, for $k \geq 3$. We then follow §9 to identify a set $\mathcal{A}$ of building blocks on $\widetilde{\Gamma}$ and a set $\mathcal{B}$ of pairwise sums of elements of $\mathcal{A}$ satisfying properties (**A**), (**B**) and (**B**$'$). To construct the master template θ on $\widetilde{\Gamma}$, with slopes as specified by $s_k(\theta)$, we proceed exactly as in §8, except that we skip the step named "Start at the First Bridge". Instead, we initialize the coefficients of the permissible functions on γ_3 in $\mathcal{B}$ so that they agree with $\varphi_C + \varphi_6$ at the midpoint of β_3. We then apply the loop subroutine on γ_3 and follow the algorithm until it terminates.

The arguments in §8 go through without change, except for Lemma 8.25. Specifically, since $2s'_2[5] = s'_2[6] + s'_2[4] = 6$, it is possible that two functions in $\mathcal{B}$ have identical slopes greater than or equal to 5 along the bridge β_3. In §8, Lemma 8.25 is used only to guarantee that no two functions assigned to the first bridge of $\widetilde{\Gamma}$ agree on that bridge, and to count the number of cohorts on the first loop. Here, we have not assigned any functions to the bridge β_3. By arguments identical to those in §8, there are at most 3 cohorts on γ_3, and at most 2 if γ_3 is skippable. We define the sets $\mathcal{S}$ and $\mathcal{T}$ exactly as in §9, and let $\mathcal{T}' = \{\psi \in \mathcal{T} : s'_2(\psi) \leq 4\}$. In each of the cases in §9, the number of functions in $\mathcal{T} \smallsetminus \mathcal{T}'$ is equal to the number of pairs (i, j) such that $s'_2[i] + s'_2[j] \geq 5$. Since there are precisely 5 such pairs, we see that $|\mathcal{T}'| = 23$. Then we show that the best approximation of the master template on $\widetilde{\Gamma}$ by $\mathcal{T}'$ is an independence on $\widetilde{\Gamma}$, exactly as in §9.

Finally, note that any function ψ that obtains the minimum on $\widetilde{\Gamma}$ satisfies $s'_2(\psi) \leq 4$. Similarly, each of the functions ψ that obtains the minimum on $\widetilde{\Gamma}'$ satisfies $s_3(\psi) \geq 5$. Since the bridge β_3 is very long, it follows that no function that obtains the minimum on one of the two subgraphs can obtain the minimum on the other. Thus, we have constructed a constructed tropical linear combination of 28 pairwise sums of functions in Σ in which 5 achieve the minimum uniquely at some point of $\widetilde{\Gamma}'$ and 23 achieve the minimum uniquely at some point of $\widetilde{\Gamma}$. In particular, this is an independence, as required. □

10.4. **Higher codimension boundary strata.** In order to verify (3), we now consider the intersection of Z with the boundary strata $\Delta_{2,j}$, each of which has codimension 2 in $\overline{\mathcal{M}}_g$.

Proposition 10.7. *The component Z does not contain any codimension 2 stratum $\Delta_{2,j}$.*

Proof. The proof is again a variation on the independence constructions from the proof of Theorem 9.1. We fix $\ell = g - j - 2$. Let Y_1 be a smooth curve of genus 2 over K whose skeleton Γ_1 is a chain of 2 loops with bridges, and let $p \in Y_1$ be a point specializing to the right endpoint of Γ_1. Similarly, let Y_2 and Y_3 be smooth curves of genus ℓ and j, respectively, whose skeletons Γ_2 and Γ_3, are chains of ℓ loops and j loops with admissible edge lengths. Suppose further that the edges in the final loop of Γ_2 are much longer than those in the first loop of Γ_3. Let $p, q \in Y_2$ be points specializing to the left and right endpoints of Γ_2, respectively, and let $q \in Y_3$ be a point specializing to the left endpoint of Γ_3. We show that $[Y'] = [Y_1 \cup_p Y_2 \cup_q Y_3] \in \Delta_{2,j}$ is not contained in Z.

As in the proof of Proposition 10.6, if $[Y'] \in Z$, then Z contains points $[X]$ corresponding to smooth curves whose skeletons are arbitrarily close to the skeleton of Y' in the natural topology on $\overline{M}_g^{\mathrm{trop}}$. In particular, there is an $X \in Z$ with skeleton a chain of loops Γ_X whose edge lengths satisfy all of the conditions in Definition 6.10, except that the bridges β_3 and β_ℓ are exceedingly long in comparison to the other edges.

Let Γ be the subgraph of Γ_X to the right of the midpoint of the bridge β_3. Note that Γ is a chain of $g-2$ loops, labeled $\gamma_3, \ldots, \gamma_g$, with bridges labeled $\beta_3, \ldots, \beta_{g+1}$.

By Lemma 10.4, there is a linear series $\ell = (L, V)$ of degree $g+3$ and rank 6 on X that is ramified at a point x specializing to the righthand endpoint v_{g+1}, and such that ϕ_ℓ is not injective. We will show that this is not possible, by adapting the tropical independence constructions from §§8-9. We define building blocks as PL functions on Γ exactly as in §8, and then, to account for the length of β_ℓ, we use the following variant on the definition of permissible functions (Definition 7.5).

Definition 10.8. Let $\psi \in \mathrm{PL}(\Gamma)$ be a function with constant slope along each bridge. We say that ψ is *ℓ-permissible* on γ_k if

- (i) either $s_j(\psi) \leq s_j(\theta)$ for all $j \leq k$, or $s_\ell(\psi) < s_\ell(\theta)$, and $s_j(\psi) \leq s_j(\theta)$ for all $\ell < j \leq k$,
- (ii) $s_{k+1}(\psi) \geq s_k(\theta)$, and
- (iii) if $s_j(\psi) < s_j(\theta)$ for some $j > k$, then $j \neq \ell$ and $s_{k'}(\psi) > s_{k'}(\theta)$ for some k' such that $k < k' < j$.

These ℓ-permissible functions behave in many ways like permissible functions. In particular, if θ is a PL function whose slopes on bridges agree with those specified for the master template, and if the best approximation of θ by ψ achieves the minimum on γ_k, then ψ must be ℓ-permissible on γ_k. Also, if ψ has constant slope along each bridge then either $s_3(\psi) > s_3(\theta)$, $s_{g+1}(\psi) < s_{g+1}(\theta)$, or ψ is ℓ-permissible on an interval of loops.

Let $\Sigma = \mathrm{trop}(V)$. Since Proposition 6.18 does not depend on the lengths of the bridges, we have that either $s'_2[5] \leq 2$ or $s'_2[4] + s'_2[6] \leq 5$. Also, since V is ramified at x, we have $s_{g+1}[6] > 6$. Even though the restriction of Σ to Γ is not the tropicalization of a linear series on a curve of genus $g-2$ with prescribed ramification at two specified points specializing to the left and right endpoints of Γ, it satisfies all of the combinatorial properties of the tropicalization of such a linear series, and we proceed to apply the arguments from §§8-9.

We construct the master template exactly as in §8, using ℓ-permissible functions in place of permissible functions. Definition 10.8 ensures that we only assign a function ψ to the left of β_ℓ if $s_\ell(\psi) \geq s_{\ell-1}(\theta)$, and we only assign it to the right of β_ℓ if $s_\ell(\psi) \leq s_{\ell-1}(\theta)$. This is precisely what is needed to make Lemmas 8.26 and 8.27 work in the present setting, where β_ℓ is very long.

Next, with the template fixed, we specify a set $\mathcal{S}$ of elements of Σ and a set $\mathcal{T}$ of pairwise sums of elements of $\mathcal{S}$, using precisely the same algorithm as in §9. In order to prove that the best approximation $\vartheta_{\mathcal{T}}$ of θ by $\mathcal{T}$ is an independence, some care must be taken to account for the length of β_ℓ, and we explain the details as follows.

The ramification conditions imply that the sum of the multiplicities of all the bridges and loops is at most 1. Hence, there are no switching bridges, and at most one switching loop. Moreover, if there is a switching loop, it has multiplicity 1, and there are no decreasing loops or bridges. Hence the choice of $\mathcal{S}$ and $\mathcal{T}$ follows either case 1 or case 3, from §9.1 or §9.3, respectively.

Among these cases, there is only one situation where the proof that $\vartheta_{\mathcal{T}}$ is an independence uses the assumption that the bridges decrease in length from left to right: when there is a loop $\gamma_{\ell'}$ that switches slope h, the interval I has length at least $m_{\ell'}$, and there are indices j and j' such that

$$s'_{\ell'-1}(\theta) = s'_{\ell'}[h] + s'_{\ell'}[j] = s'_{\ell'}[h] + s'_{\ell'}[j'] + 1.$$

In this situation, we must show that the best approximation of θ by $\varphi_C + \varphi_j$ achieves equality on the region where $\varphi^0_{h+1} + \varphi_j$ achieves the minimum, and the best approximation of θ by $\varphi_A + \varphi_j$ achieves equality on the region where $\varphi^\infty_h + \varphi_j$ achieves the minimum. In Lemma 9.17 and related arguments, this is done by noting that both functions have slope larger than that of θ on intervals of length $t \geq m_{\ell'}$. In the present case, this is insufficient, because we may have $\ell' < \ell$, and the bridge β_ℓ is very long.

However, since there are no decreasing loops or bridges, we have

$$s_k[h] + s_k[j] \geq s'_{\ell'}[h] + s'_{\ell'}[j] \text{ for all } k \geq \ell'.$$

It follows that $s_k(\varphi_A + \varphi_j) \geq s_k(\theta)$ and $s_k(\varphi_C + \varphi_j) \geq s_k(\theta)$ for all $k \geq \ell'$, and the result follows. Therefore, the construction yields an independence among 28 pairwise sums of functions in Σ, and the proposition follows. □

Propositions 10.3, 10.6, and 10.7 show that Z satisfies conditions (1)-(3) in Proposition 2.2. For $g = 23$, we conclude that $\mathfrak{U} \subseteq \widetilde{\mathcal{G}}^6_{26}$ is generically finite over each codimension one component of its image in $\overline{\mathcal{M}}_{23}$, and hence $\overline{\mathcal{M}}_{23}$ is of general type.

For $g = 22$, we proceed to verify (4) by studying the pull back of Z to $\overline{\mathcal{M}}_{3,1}$.

10.5. **Pulling back to $\overline{\mathcal{M}}_{3,1}$.** Recall that $\jmath_3 : \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}_g$ is the map obtained by attaching a fixed general pointed curve of genus $g-3$ to an arbitrary stable pointed curve of genus 3.

Proposition 10.9. *The preimage $\jmath_3^{-1}(Z)$ is contained in the union of the Weierstrass locus $\overline{\mathcal{W}}_3$ and the hyperelliptic locus $\overline{\mathcal{H}}_3$ in $\overline{\mathcal{M}}_{3,1}$.*

We prove this proposition using a variation of the arguments from the vertex avoiding case in §7, as follows. Let X be a curve of genus 19 over K whose skeleton Γ is a chain of 19 loops with bridges, with admissible edge lengths. Let $q \in X$ be a point specializing to the left endpoint w_0 of Γ, and let $\jmath_3 : \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}_g$ be the map obtained by attaching an arbitrary stable pointed curve (X', q) of genus 3 to (X, q). We now show that the curve $[Y] = [X \cup_q X']$ is not in Z when X' is not hyperelliptic and $q \in X'$ is not a Weierstrass point.

As in Lemma 10.5, if $[Y] \in Z$, then there is a limit linear series ℓ of degree 25 and rank 6 on Y such that the multiplication map on each aspect of ℓ is not injective. Let $\ell_X \subseteq H^0(X, \mathcal{O}(D_X))$ be the X-aspect of such a limit linear series. As in Lemma 10.4, we may assume that ℓ_X is ramified at a point p specializing to the right endpoint v_{20} of Γ. To complete the proof of the proposition, we use a variation on the arguments from §7 to show that there are 28 tropically independent pairwise sums of functions in $\Sigma := \mathrm{trop}(\ell_X)$.

We may assume that $D = \mathrm{Trop}(D_X)$ is a break divisor. We claim that

$$\sum_{i=0}^{3} a_i^{\ell_X}(q) \geq 13.$$

Since X' is not hyperelliptic, we have $a_{r-1}^{\ell_{X'}}(q) \leq d-3$. Furthermore, if equality holds, then since $q' \in X'$ is not a Weierstrass point, we have $a_r^{\ell_{X'}}(q) \leq d-1$. The claim then follows from the definition of a limit linear series.

By (29), it follows that $\sum_{i=3}^{6} s'_0[i] \leq 9$. The ramification condition at the point specializing to v_{20} implies $s_{20}[6] \geq 7$. By Proposition 6.18, it follows that all of the bridges and loops have multiplicity zero, and the inequalities on slopes must in fact be equalities:

$$\sum_{i=3}^{6} s'_0[i] = 9, \text{ and } s_{20}[6] = 7.$$

Because of this, we treat this case in a similar manner to the vertex avoiding case of §7. There are finitely many such classes; they are in bijection with standard Young tableaux on one of the three shapes depicted in Figure 27. The particular shape is determined by the sequence of slopes along the first bridge β_1. More precisely, the three missing boxes from the upper left corner form the partition λ'_0. That this partition consists of precisely 3 boxes corresponds to the fact that $\sum_{i=3}^{6} s'_0[i] = 9$. We refer to these three shapes as Case A, Case B, and Case C, respectively.

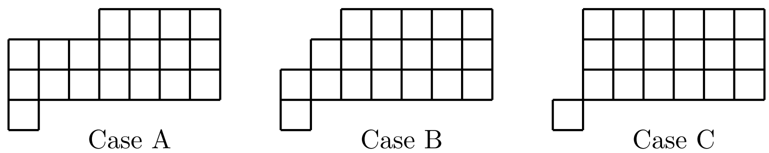

FIGURE 27. Skew Partitions Corresponding to the Given Slope Conditions

In Case C, when $s'_0[6] = 6$ and $s'_0[i] = i - 3$ for all $i < 6$, we may employ a strategy similar to that of Remark 7.23. Specifically, we see that $s_1[i] + s_1[j] \geq 5$ if and only if one of i or j is equal to 6, and there are exactly two pairs (i, j) such that $s_1[i] + s_1[j] = 4$. We may therefore build an independence among the functions φ_{ij} by assigning the functions $\varphi_{66}, \varphi_{56}, \varphi_{46}, \varphi_{36}$, and φ_{26} to the first bridge β_1, and then proceeding according to our algorithm from §7. This works because the listed functions all have distinct slopes greater than 4 along the first bridge.

In the other two cases, however, we see that

$$s_1[4] + s_1[6] = 2s_1[5] = 6 \geq 5,$$

so this strategy does not work. To handle these cases, we provide a variation on the algorithm presented in §7. Rather than dividing the graph Γ into 3 blocks, we instead divide it into 5 blocks, as follows. Let z_1 be the smallest symbol appearing in the first two rows of the tableau. (Note that, because this is a skew tableau, z_1 is not necessarily equal to 1.) Similarly, let z_2 be the second smallest symbol in the first two rows of the tableau. In Case A, let z_3 be the 4th smallest symbol appearing in the union of the first and third row, and in Case B, let z_3 be the 5th smallest symbol appearing in the union of the first and third row. Finally, in Case A, let z_4 be the 9th smallest symbol appearing in the union of the second and third row, and in Case B, let z_4 be the 8th smallest symbol appearing the union of the second and third row.

The incoming slopes of θ at v_k, the leftmost point on γ_k, will be:

$$s_k(\theta) = \begin{cases} 6 & \text{if } k \leq z_1, \\ 5 & \text{if } z_1 < k \leq z_2, \\ 4 & \text{if } z_2 < k \leq z_3, \\ 3 & \text{if } z_3 < k \leq z_4, \\ 2 & \text{if } z_4 < k \leq 19. \end{cases}$$

We now count the number of permissible functions on each block, as in §7. We first show the following.

Lemma 10.10. *For any loop γ_k, there are at most 3 non-departing permissible functions on γ_k. Moreover, there are at most 3 permissible functions on the loops γ_1, γ_{z_1+1}, and γ_{z_4+1}.*

Proof. The proof of Lemma 7.18 holds in all cases, except when γ_k is contained in the third block. In this case, if there are 4 non-departing permissible functions on γ_k, then as in the proof of Lemma 8.23, we must have

$$s_{k+1}(\varphi_i) + s_{k+1}(\varphi_{6-i}) = 4 \text{ for all } i.$$

In other words, if we consider the skew tableau consisting of symbols less than or equal to k, we see that the sum of heights of the ith column and the $(6 - i)$th column must be equal to 4. We therefore see that k must be greater than z_3, a contradiction. □

We now define 3 more loops. Each will be contained in one of the last three blocks. Let b_3 be the third smallest symbol in the first two rows of the tableau. In Case A, let b_4 be the 5th smallest symbol appearing in the union of the first and third row, and in Case B, let b_4 be the 6th smallest symbol appearing in the union of the first and third row. Finally, in Case A, let b_5 be the 10th smallest symbol appearing the union of the second and third row, and in Case B, let b_5 be the 9th smallest symbol appearing in the union of the second and third row. Note the following inequalities:

$$z_1 < z_2 < b_3 < z_3 < b_4 < z_4 < b_5.$$

Lemma 10.11. *There are no new permissible functions on the loops γ_{z_3} and γ_{z_4}. If γ_ℓ is not the first loop in a block, for $\ell \in \{z_1, z_2, b_3, b_4, b_5\}$, then there are no new permissible functions on γ_ℓ. If either γ_{b_3} or γ_{b_4} is the first loop in a block, then there are only 3 permissible functions on it. If $\gamma_{z_1}, \gamma_{z_2}$, or γ_{b_5} is the first loop in a block, then there are only 2 permissible functions on it. If $k \neq z_i$ or b_i for any i, then there is a new permissible function on γ_k.*

Proof. The proof is identical to that of Lemma 7.15. For each of these loops γ_ℓ, first enumerate the possible sequences of slopes $s_{\ell+1}[i]$. Then note that, for any value i that could satisfy $s_{\ell+1}[i] > s_\ell[i]$, there is no value j such that $s_{\ell+1}[i] + s_{\ell+1}[j] = s_\ell(\theta)$. Such values of i must necessarily satisfy $s_{\ell+1}[i] > s_{\ell+1}[i-1]+1$, but the converse is not true. For example, we consider the case $\ell = z_1$, and leave the remaining cases to the interested reader. The possible sequences of slopes $s_{z_1+1}[i]$ are:

$$(-3,-2,-1,1,2,3,4)$$

$$(-3,-2,-1,0,2,3,5)$$

$$(-3,-2,-1,0,1,4,5).$$

$$(-3,-2,-1,0,1,4,6).$$

$$(-3,-2,-1,0,1,4,7).$$

By the definition of z_1, in the last two cases we have $s_{z_1+1}[6] = s_{z_1}[6]$. In each of the cases, we see that for any of the remaining values of i satisfying $s_{\ell+1}[i] > s_{\ell+1}[i-1]+1$, there is no value j such that $s_{\ell+1}[i] + s_{\ell+1}[j] = 6$.

We will prove the last statement in the case where none of the loops listed above are the first loop in a block. The other cases are similar. Note that there are 2 functions φ_{ij} satisfying $s_1(\varphi_{ij}) > 6$, and two more functions φ_{ij} satisfying $s_{20}(\varphi_{ij}) < 2$. Each of the remaining 24 functions is permissible on some loop. Of these, the number of functions that are new on the first loop of a block is at most $3+3+4+4+3 = 17$, leaving at least 7 functions. Three of the blocks contain a loop with no permissible functions, and the other two blocks contain two such loops. There are therefore $19-7-5 = 7$ remaining loops that are not the first loop in a block. Since the number of functions remaining is greater than or equal to the number of loops remaining, we see that we must in fact have equality, and there must be a new function on each of these loops. □

Corollary 10.12. *On each of the 5 blocks, the number of permissible functions is 1 more than the number of loops.*

Proof of Proposition 10.9. Lemma 10.10 and Corollary 10.12 allow us to prove the proposition by running the algorithm from §7. More specifically, when we run the algorithm, we assign a function to each loop, and one extra function to the bridge at the end of each block. There are only 19 loops in this case, but there are 5 blocks instead of 3, so we assign 19+5=24 functions in this way, plus 2 more on the first bridge β_1, and 2 more on the last bridge β_{20}. We therefore obtain an independence among 28 functions on Γ. This completes the proof of Proposition 10.9. □

Propositions 10.3, 10.6, 10.7, and 10.9 together show that Z satisfies the vanishing conditions from Proposition 2.2. We conclude that the degeneracy locus Z is generically finite over each codimension 1 component of its image. This proves Theorem 1.4 and completes the proof of Theorem 1.1. □

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