CHIP FIRING

12. Brill-Noether Theory and Gonality

In this lecture, we discuss several open problems concerning the existence and behavior of divisors on graphs of given degree and rank. In the previous lecture, we classified divisors of a given rank and degree on a chain of loops. The existence of such divisors is governed by the *Brill-Noether number*

$$\rho(g,r,d) := g-(r+1)(g-d+r).$$

In particular, if $\rho(g,r,d) \geq 0$, then *every* chain of g loops possesses a divisor of degree d and rank at least r. It is natural to ask whether this conclusion holds for graphs more generally.

Question 12.1. *If $\rho(g,r,d) \geq 0$, does every graph of genus g possess a divisor of degree d and rank at least r?*

While the Brill-Noether existence problem remains open, there is a slight variant of it that is known to hold. To understand this variant, we first need a definition.

Definition 12.2. *Let G be a graph and m a positive integer. We define the mth* refinement *of G to be the graph obtained by subdividing each edge of G into m edges.*

Theorem 12.3. *Let G be a graph of genus g. If $\rho(g,r,d) \geq 0$, then there exists a positive integer m such that the mth refinement of G possesses a divisor of degree d and rank r.*

In order to prove Theorem 12.3, we will need techniques from algebraic geometry. As of the time of writing, there is no known proof of Theorem 12.3 using purely combinatorial methods. The Brill-Noether existence problem asks whether the integer m in Theorem 12.3 can be taken to be 1.

We might try to quantify the Brill-Noether existence problem by defining certain graph invariants.

Definition 12.4. *Let G be a graph. The* gonality *of G is defined to be*

$$\mathrm{gon}(G) := \min\{\deg(D) \mid \mathrm{rk}(D) \geq 1\}.$$

The Clifford index *of G is defined to be*

$$\mathrm{Cliff}(G) := \min\{\deg(D) - 2\,\mathrm{rk}(D) \mid \mathrm{rk}(D) \geq 1 \text{ and } \mathrm{rk}(K_G - D) \geq 1\}.$$

By definition, we have

$$\mathrm{Cliff}(G) \leq \mathrm{gon}(G) - 2.$$

Results from algebraic geometry suggest that we should also have

$$\mathrm{Cliff}(G) \geq \mathrm{gon}(G) - 3.$$

Date: February 25, 2019.

As of the time of writing, the validity of this latter inequality remains open. The following question is a subcase of the Brill-Noether existence problem.

Question 12.5. *Does every graph of genus g have gonality at most $\lfloor \frac{g+3}{2} \rfloor$?*

In the previous lecture, we also saw that, if $\rho(g,r,d) < 0$, then there exists a chain of g loops that does *not* possess a divisor of degree d and rank at least r. It is interesting to ask about the *typical* behavior. If we pick a graph at random, with what probability should we expect that graph to possess a divisor of given degree and rank? What is the expected gonality or Clifford index of a random graph? Of course, such questions depend on what we mean by a "random graph". There are several different models of random graphs in the literature, and the expected value of these graph invariants depends on the model that we choose.

The current best known results on the gonality of random graphs rely on bounding the gonality in terms of other graph invariants. One invariant that has attracted a great deal of attention in the extremal combinatorics community is the *treewidth*. There are several different definitions of treewidth, but we will define it in terms of *brambles*.

Definition 12.6. *Let G be a graph. A* bramble *on G is a collection $\mathcal{B}$ of subsets of $V(G)$ satisfying:*

(1) *For every $B \in \mathcal{B}$, the induced subgraph $G[B]$ is connected, and*
(2) *For every pair $B, B' \in \mathcal{B}$, the induced subgraph $G[B \cup B']$ is connected.*

Definition 12.7. *Let $\mathcal{B}$ be a bramble on a set G. A subset $S \subseteq V(G)$ is a* hitting set *for $\mathcal{B}$ if $S \cap B \neq \emptyset$ for all $B \in \mathcal{B}$. The* order *of $\mathcal{B}$, $\|\mathcal{B}\|$, is the minimum size of a hitting set for $\mathcal{B}$.*

Before we explore the connection between brambles and gonality, we need the following preliminary lemma.

Lemma 12.8. *Let G be a graph, $U \subset V(G)$, and $\mathcal{B}$ a bramble on G. Suppose there exist $B, B' \in \mathcal{B}$ such that $B \subseteq U$ and $B' \subseteq U^c$. Then*

$$\|\mathcal{B}\| \leq |E(U, U^c)| + 1.$$

Proof. Let $X \subseteq U$ be the set of vertices in U with a neighbor in U^c, and $Y \subseteq U^c$ be the set of vertices in U^c with a neighbor in U. Without loss of generality, we may assume that there is no $B'' \in \mathcal{B}$ such that $B'' \subseteq U$ and $B'' \cap X$ is strictly contained in $B \cap X$.

We now construct a hitting set S for $\mathcal{B}$ of size $|E(U, U^c)| + 1$. First, let v be a vertex in $B \cap X$. Next, for each edge $e \in E(U, U^c)$, define the vertex v_e as follows. If e has an endpoint in B, then v_e is the endpoint of e in Y. Otherwise, v_e is the endpoint in X. We define

$$S = \{v\} \cup \{v_e | e \in E(U, U^c)\}.$$

To see that S is a hitting set for $\mathcal{B}$, let $A \in \mathcal{B}$. If $A \subset U$, then by our assumption on B, either $B \cap X \subseteq A \cap X$ or there exists a vertex $x \in A \cap X$ that is not contained in B. In the first case, $v \in S \cap A$, and in the second case, $x \in S \cap A$.

If $A \subset U^c$, then since $\mathcal{B}$ is a bramble, there exists an edge with one endpoint in B and the other endpoint in A. By construction, the endpoint in A is contained in S.

Finally, if A intersects both U and U^c, then since A is connected, there exists an edge $e \in E(U, U^c)$ with both endpoints in A. By construction, one of the endpoints of e is in S. It follows that S is a hitting set, hence

$$\|\mathcal{B}\| \leq |S| = |E(U, U^c)| + 1.$$

□

Example 12.9. Let T be a tree and $\mathcal{B}$ a bramble on T. If $B, B' \in \mathcal{B}$ are disjoint, then there exists a set $U \subset V(T)$ with $B \subseteq U$, $B' \subseteq U^c$, and $|E(U, U^c)| = 1$. By Lemma 12.8, it follows that $\|\mathcal{B}\| = 2$.

On the other hand, suppose that every pair of sets $B, B' \in \mathcal{B}$ have nonempty intersection. We show, by induction on $|V(T)|$, that there exists a vertex $v \in V(T)$ with $v \in B$ for all $B \in \mathcal{B}$. Equivalently, we will see that $\|\mathcal{B}\| = 1$. To see this, let $v \in V(T)$ be a leaf. If $v \in B$ for all $B \in \mathcal{B}$, then we are done. Otherwise, by assumption we cannot have $\{v\} \in \mathcal{B}$. Therefore, every set B in the bramble $\mathcal{B}$ that contains v also contains the unique vertex adjacent to v. If T' is the tree obtained by removing the leaf v from $V(T)$, then

$$\mathcal{B}' = \{B \cap V(T') | B \in \mathcal{B}\}$$

is a bramble on T' with the property that every pair of sets have nonempty intersection. By induction, therefore, there is a vertex $v \in V(T)$ contained in every set of the bramble $\mathcal{B}$.

Taken together, we see that every bramble $\mathcal{B}$ on a tree T satisfies $\|\mathcal{B}\| \leq 2$.

We now describe the connection between brambles and gonality.

Theorem 12.10. *Let G be a graph and $\mathcal{B}$ a bramble on G. Then*

$$\mathrm{gon}(G) \geq \|\mathcal{B}\| - 1.$$

Proof. Let D be an effective divisor of positive rank. It suffices to show that

$$\deg(D) \geq \|\mathcal{B}\| - 1.$$

Suppose that $\deg(D) < \|\mathcal{B}\|$. By definition, no effective divisor of degree $\deg(D)$ is a hitting set for the bramble $\mathcal{B}$. Without loss of generality, we may assume that $\mathrm{Supp}(D)$ intersects the maximum number of sets in $\mathcal{B}$, over all effective divisors equivalent to D. Since $\mathrm{Supp}(D)$ is not a hitting set for $\mathcal{B}$, there exists a set $B \in \mathcal{B}$ such that $B \cap \mathrm{Supp}(D) = \emptyset$.

Let $v \in B$. By Dhar's burning algorithm, there exists a sequence of subsets

$$U_1 \subseteq U_2 \subseteq \ldots \subseteq U_k \in V(G) \smallsetminus \{v\}$$

such that, if D_i is the divisor obtained from D by firing sets $U_1, \ldots, U_i$, then D_i is effective, and D_k is v-reduced. Since D has positive rank, $D_k(v) > 0$.

Let $j \leq r$ be the minimum value such that $\mathrm{Supp}(D_j)$ has nonempty intersection with B. Since $D_k(v) > 0$, such a j exists. Since $\mathrm{Supp}(D)$ intersects the maximum number of sets in $\mathcal{B}$, we see that there exists a set $B' \in \mathcal{B}$ such that $\mathrm{Supp}(D) \cap B' \neq \emptyset$

and $\mathrm{Supp}(D_j) \cap B' = \emptyset$. It follows that $B' \subseteq U_j$ and $B \subseteq U_j^c$. By Lemma 12.8, we have

$$\|\mathcal{B}\| \leq |E(U_j, U_j^c)| + 1.$$

On the other hand, Dhar's burning algorithm gives us

$$\deg(D) \geq |E(U_j, U_j^c)| \geq \|\mathcal{B}\| - 1.$$

□

Remark 12.11. The *treewidth* of a graph G is 1 less than the maximum order of a bramble on G. An alternative formulation of Theorem 12.10 is therefore that the gonality of a graph G is bounded below by its treewidth.